Analytic theory of difference equations with rational and elliptic coefficients and the Riemann-Hilbert problem

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Abstract

A new approach to the analytic theory of difference equations with rational and elliptic coefficients is proposed. It is based on the construction of canonical meromorphic solutions which are analytical along "thick paths". The concept of such solutions leads to the notion of local monodromies of difference equations. It is shown that in the continuous limit they converge to the monodromy matrices of differential equations. New type of isomonodromic deformations of difference equations with elliptic coefficients changing the periods of elliptic curves is constructed.

1 Introduction

It is well-known that correlation functions of diverse statistical models, gap probabilities in the Random Matrix Theory can be expressed in terms of solutions of the Painlevé type *differential* equations (see [1, 2, 3, 4, 5] and references therein). In the recent years discrete analogs of the Painlevé equations [6, 7] have attracted considerable interest due to their connections to discrete probabilistic models [8, 9]. In [10] it was found that the general setup for these equations is provided by the theory of *isomonodromy* transformations of linear systems of difference equations with rational coefficients.

The analytic theory of matrix linear difference equations

$$\Psi(z+1) = A(z)\Psi(z) \tag{1.1}$$

with rational coefficients is a subject of its own interest. It goes back to the fundamental results of Birkhoff [11, 12] which have been developed later by many authors (see the book [13] and references therein).

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Difference equations (1.1) are classified in a rough way by terms: *regular, regular singular, mild and wild* (see [13] for details). The terminology reflects the formal asymptotic theory of the equation near the infinity. Equation (1.1) with the coefficients of the form

$$A = A_0 + \sum_{m=1}^{n} \frac{A_m}{z - z_m} \tag{1.2}$$

is regular singular if $A_0 = 1$. It is regular if in addition A has no residue at the infinity, i.e. $\sum_{m=1}^{n} A_m = 0$. The mild equations are those for which the matrix A_0 is invertible. In this paper we restrict ourself to the case of mild equations with a diagonalizable leading coefficient A_0. It will be assumed also that the poles z_m are not congruent, i.e. $z_l - z_m$ is not an integer, $z_l - z_m \notin Z$.

Equation (1.1) is invariant under the transformation $\Psi' = \rho^z \Psi,\ A' = \rho A(z)$, where ρ is a scalar. It is also invariant under the gauge transformation $\Psi' = g\Psi,\ A' = gA(z)g^{-1},\ g \in SL_r$. Therefore, if A_0 is diagonalizable, then we may assume without loss of generality that A_0 is a diagonal matrix of determinant 1,

$$A_0^{ij} = \rho_i \delta^{ij}, \quad \det A_0 = \prod_j \rho_j = 1. \tag{1.3}$$

In addition, throughout the paper it will be assumed that

$$\mathrm{Tr}\,(\mathrm{res}_\infty A dz) = \mathrm{Tr}\left(\sum_{m=1}^{n} A_m\right) = 0. \tag{1.4}$$

If the eigenvalues of A_0 are pairwise distinct $\rho_i \neq \rho_j$, then equation (1.1) has a unique formal solution $Y(z)$ of the form

$$Y = \left(1 + \sum_{s=1}^{\infty} \chi_s z^{-s}\right) e^{z \ln A_0 + K \ln z}, \tag{1.5}$$

where $K^{ij} = k_i \delta^{ij}$ is a diagonal matrix.

In [11, 12] difference equations with *polynomial* coefficients $\widetilde{A}$ were considered. Note, that the general case of rational $A(z)$ is reduced to the polynomial one by the transformation

$$\widetilde{A} = A(z) \prod_m (z - z_m), \quad \widetilde{\Psi} = \Psi \prod_m \Gamma(z - z_m), \tag{1.6}$$

where $\Gamma(z)$ is the Gamma-function. Birkhoff proved that, if the ratios of eigenvalues ρ_i of the leading coefficient of $\widetilde{A}$ are not real, $\Im\,(\rho_i/\rho_j) \neq 0$, then equation (1.1) with polynomial coefficients has two canonical meromorphic solutions $\widetilde{\Psi}_r(z)$ and $\widetilde{\Psi}_l(z)$ which are holomorphic and asymptotically represented by $\widetilde{Y}(z)$ in the half-planes $\Re z >> 0$ and $\Re z << 0$, respectively. Moreover, Birkhoff proved that the connection matrix

$$\widetilde{S}(z) = \widetilde{\Psi}_r^{-1}(z)\widetilde{\Psi}_l(z)\,, \tag{1.7}$$

which must be periodic for obvious reason, is, in fact, a rational function in $\exp(2\pi i z)$. This function has just as many constants involved as there are parameters in $\widetilde{A}$. The other result

of Birkhoff implies that if two polynomial matrix functions $\widetilde{A}'(z)$ and $\widetilde{A}(z)$ have the same connection matrix $S(z)$ then there exists a rational matrix $R(z)$ such that

$$\widetilde{A}'(z) = R(z+1)\widetilde{A}(z)R^{-1}(z)\,. \tag{1.8}$$

In [10] a family of commuting transformations (1.8) was explicitly constructed. Furthermore, it was shown that in the continuous limit the commutativity equations for a certain subset of these transformations converge to the classical Schlesinger equations ([14]).

Until now key ideas of Birhoff's approach to the analytic theory of difference equations have remained intact. A construction of actual solutions of (1.1) having prescribed asymptotic behavior in various sectors at infinity resembles rather the Stocks' theory of differential equations with irregular singularities, then the conventional theory of differential equations with regular singularities. The monodromy representation of $\pi_1(C\backslash\{z_1,\ldots,z_n\})$ which provides the integrals of motion for the Schlesinger equations, has no obvious analog in discrete situation. On the other hand the obvious differential analog of the connection matrix $S(z)$ gives only the monodromy information at infinity and provides no information on local monodromies around the poles z_m. (Maybe by this reason Birkhoff just from the beginning eliminated the positions of poles and restricted himself to the case of polynomial coefficients).

The main goal of this paper is to develop a new approach to the analytic theory of difference equations with rational coefficients and extend it to the case of equations with *elliptic* coefficients. It is based on the construction of meromorphic solutions of difference equations which are holomorphic along *thick paths*.

It is instructive to present the case just opposite to the Birkhoff's one, namely, the case of real *exponents* ρ_i. Let x be a real number such that $x \neq \Re z_i$. Consider matrix solution $\Psi_x(z)$ of equation (1.1) that is non-degenerate and holomorphic inside the strip $z \in \Pi_x : x \leq \Re z \leq x+1$ and continuous up to the boundary. It is also required that in Π_x the solution Ψ_x grows at most *polynomially* as $|\Im z| \to \infty$. It is easy to show that if such a solution exists then it is unique up to the transformation $\Psi'_x = \Psi_x(z)g,\ \ g \in GL_r$. Moreover, it turns out that, if Ψ_x exists, then it has the following asymptotic representation

$$\Psi_x = Y g_x^{\pm}, \quad \Im z \to \pm\infty\ . \tag{1.9}$$

To some extend, the ratio

$$g_x = g_x^{+}\left(g_x^{-}\right)^{-1} \tag{1.10}$$

can be regarded as a transfer matrix of the solution along the "thick" path Π_x from $-i\infty$ to $i\infty$.

Furthermore, we show that for $x >> 0$ and $x << 0$ the solution Ψ_x *does exist*. In these regions it is x-independent. Therefore, we get two meromorphic solutions Ψ_r and Ψ_l of equation (1.1), which are holomorphic in the half-planes $\Re z >> 0$ and $\Re z << 0$, respectively. The corresponding transfer matrices $g_r = g_x$, $x >> 0$, and $g_l = g_x$, $x << 0$ are "quasi"-upper or -lower triangular matrices, i.e.

$$g_{r(l)}^{ii} = 1,\ g_r^{ij} = 0,\ \text{if}\ \rho_i < \rho_j, \quad g_l^{ij} = 0,\ \text{if}\ \rho_i > \rho_j, \tag{1.11}$$

This result clarifies the well-known fact that, if $\Im(\rho_i/\rho_j) = 0$, then there are no Birkhoff's solutions with uniform asymptotic representation in the half-planes $\Re z >> 0$ and $\Re z << 0$.

If the solutions Ψ_r, Ψ_l are normalized by the condition $g_x^- = 1$, then their connection matrix has the form

$$S(z) = \Psi_r^{-1}(z)\Psi_l(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \tag{1.12}$$

where

$$S_\infty = 1 + \sum_{m=1}^{n} S_m = g_r^{-1} e^{2\pi i K} g_l \,, \tag{1.13}$$

and K is the same diagonal matrix as in (1.5). To the best of the author's knowledge, the explicit form (1.12) of the connection matrix including the relations (1.11,1.13) is a new result even for the case of regular singular equations for which $g_{r(l)} = 1$ (compare it with the Theorem 10.8 in [13]).

The direct monodromy map

$$A(z) \to S(z) \tag{1.14}$$

for regular singular and mild equations is constructed in sections 2 and 3, respectively. In section 2.2 we introduce a notion of *local monodromies* of difference equations. First, they are defined for three examples of regular singular equations. Namely, for the case of special equations with the coefficients $A \in \mathcal{A}_0$ of the form (1.2) such that $\det A(z) \equiv 1$. The second case considered is the case of unitary difference equations with the coefficients $A \in \mathcal{A}^U$ satisfying the relation $A^+(\bar{z}) = A^{-1}(z)$. The third example is the small norm case, i.e. the case of equations with coefficients such that $|A_m| < \varepsilon$.

The existence of the canonical solution Ψ_x is equivalent to solvability of an auxiliary system of linear singular equations. The index of that system equals

$$ind_x A = \frac{1}{2\pi i} \int_L d \ln \det A, \quad z \in L : \Re z = x. \tag{1.15}$$

Fundamental results of the theory of singular integral equations ([15]) imply that, if $ind_x A = 0$ then for generic A the canonical solution Ψ_x exists.

The index $ind_x A$ vanishes identically, if $\det A = 1$. Therefore, for generic $A \in \mathcal{A}_0$ the solution Ψ_x exists for all $x \neq \Re z_k$. It is *x-independent*, when x varies between the values $\Re z_k$. Suppose that $\Re z_1 < \Re z_2 < \cdots < \Re z_n$, then we obtain a set of $n+1$ meromorphic solutions $\Psi_k(z)$ of equation (1.1) that are holomorphic in the domains $\Re z_k < \Re z < \Re z_{k+1} + 1$ (here $k = 0, \ldots, n$, and for brevity we formally set $z_0 = -\infty$ and $z_{n+1} = \infty$).

The local connection matrices $M_k = \Psi_k^{-1}\Psi_{k-1}$ have the form

$$M_k = 1 - \frac{m_k}{e^{2\pi i(z-z_k)} - 1}. \tag{1.16}$$

The evaluation of M_k at $z = i\infty$ equals

$$\mu_k = 1 + m_k = g_k^{-1} g_{k-1}, \tag{1.17}$$

where g_k is the transfer matrix (1.10) along the strip Π_x for $\Re z_k < x < \Re z_{k+1}$. The matrix μ_k is a discrete analog of the *monodromy matrix* along a path from $-i\infty$ which goes around the puncture z_k and returns back to $-i\infty$.

The monodromy matrices μ_k uniquely define local connection matrices $M_k(z)$ and the global connection matrix (1.12), which is equal to the product

$$S(z) = M_n(z)M_{n-1}(z)\cdots M_1(z). \tag{1.18}$$

Note, that a generic unimodular matrix $S(z)$, $\det S = 1$, of the form (1.12) has a unique representation (1.18), where the factors M_k have the form (1.16). Therefore, the correspondence $S(z) \leftrightarrow \{\mu_k\}$ is one-to-one on open sets of the corresponding spaces.

In all the three examples of difference equations considered in section 2.2, we show that the monodromy map (1.14) is one-to-one on open sets of the corresponding spaces. The solution of the inverse monodromy problem of reconstruction of the coefficients $A(z)$ from the monodromy data is reduced to a certain Riemann-Hilbert factorization problem on a set of vertical lines. The possibility of this reduction is based on an existence of intermediate solutions $\Psi_l, \Psi_1, \ldots, \Psi_r$, whose domains of analyticity overlap and cover the whole complex plane.

For generic difference equations the inverse monodromy problem is solved in section 3. We prove that the monodromy map restricted to the subspace $\mathcal{A}_D$ of coefficients having *fixed* determinant

$$A \in \mathcal{A}_D \subset \mathcal{A}: \quad \det A(z) = D(z) = \frac{\prod_{\alpha=1}^N (z-\zeta_\alpha)}{\prod_{i=1}^n (z-z_m)^{h_m}}, \quad N = \sum_m h_m\,, \tag{1.19}$$

is a one-to-one correspondence of open dense sets. If the zeros ζ_α of D are not congruent to each other, then the injectivity of (1.14) restricted to $\mathcal{A}_D$ follows directly from the construction of canonical solutions. Isomonodromy transformations are used as an important intermediate step for the proof of surjectivity of (1.14).

Let us call two rational functions D and D' of the form (1.19) equivalent if their zeros and poles are pairwise congruent, i.e. $\zeta_\alpha - \zeta'_\alpha \in Z$, $z_m - z'_m \in Z$. It turns out that for each pair of equivalent functions there exists a birational isomonodromy isomorphism $T_D^{D'} : \mathcal{A}_D \longmapsto \mathcal{A}'_D$. Therefore, in order to prove that there is a map

$$\mathcal{S}_{\widehat{D}} \longmapsto \mathcal{A}_D, \tag{1.20}$$

which is inverse to the restriction of (1.14) to $\mathcal{A}_D$, it is enough to construct (1.20) for at least one D in each equivalence class $[D]$. Here $\mathcal{S}_{\widehat{D}}$ is the space of connection matrices having fixed determinant $\widehat{D} = D(w)$, $w = e^{2\pi i z}$; and (1.20) is defined on an open dense subspace of $\mathcal{S}_{\widehat{D}}$.

In each equivalence class $[D]$ there exists a representative D such that its zeros and poles belong to Π_x. In that case the canonical meromorphic solutions Ψ_l and Ψ_r are holomorphic in the domains $\Re z < x+1$ and $\Re z > x$, which overlap. Then, the problem of reconstruction of $\Psi_{r(l)}$ is reduced to the standard Riemann-Hilbert factorization problem on the line $\Re z = x + 1/2$.

In Section 4 we consider the continuous limit of our construction. It turns out that the canonical meromorphic solutions Ψ_x of a difference equation

$$\Psi(z+h)=\left(1+hA_0+h\sum_{m=1}^{n}\frac{A_m}{z-z_m}\right)\Psi(z) \tag{1.21}$$

exist for any x such that $|x-\Re z_m|>Ch$. Furthermore, we show that in the limit $h\to 0$ this solution in the neighborhood of the path $\Re z=x$ converges to a solution of the differential system

$$\frac{d\widehat{\Psi}}{dz}=\left(A_0+\sum_{m=1}^{n}\frac{A_m}{z-z_m}\right)\widehat{\Psi}(z). \tag{1.22}$$

That implies that the monodromy matrices μ_k do converge to the conventional monodromy matrices of the corresponding system of differential equations. For difference equations with real exponents the transfer matrices $g_{r(l)}$ converge to the Stokes' matrices of equation (1.22) at the infinity, where the differential equation (1.22) has irregular singularity. Similar result is obtained for the Birkhoff's case of imaginary exponents.

In Section 5 we extend our consideration to the case of difference equations with "elliptic" coefficients. More precisely, we consider the equation

$$\Psi(z+h)=A(z)\Psi(z), \tag{1.23}$$

where $A(z)$ is a meromorphic $(r\times r)$ matrix function with simple poles, which satisfies the following monodromy properties

$$A(z+2\omega_\alpha)=B_\alpha A(z)B_\alpha^{-1},\quad B_\alpha\in SL_r,\ \alpha=1,2. \tag{1.24}$$

The matrix $A(z)$ can be seen as a meromorphic section of the vector bundle $Hom(\mathcal{V},\mathcal{V})$, where $\mathcal{V}$ is a holomorphic vector bundle on the elliptic curve Γ with periods $2\omega_\alpha$, which is defined by a pair of commuting matrices B_α. If B_α are diagonalizable then without loss of generality we may assume that $\mathcal{B}_\alpha$ are diagonal. Furthermore, using the gauge transformations defined by diagonal matrices of the form G^z one can make B_1 to be equal to the identity matrix. In this gauge the second matrix B_2 can be represented in the form $B_2=e^{\pi i\hat{q}/\omega_1}$, where $\hat{q}$ is a diagonal matrix.

Without loss of generality we may assume that $\Im(\omega_2/h)>0$. Along the lines identical to that in the rational case, we define the canonical meromorphic solutions Ψ_x of equation (1.23). They satisfy the following *Bloch* monodromy property

$$\Psi_x(z+2\omega_2)=e^{\pi i\hat{q}/\omega_1}\Psi_x(z)e^{-2\pi i\hat{s}/h}, \tag{1.25}$$

where $\hat{s}$ is a diagonal matrix $\hat{s}^{ij}=s_i\delta^{ij}$. The connection matrix S_x of two such solutions $\Psi_x(z)$ and $\Psi_{x+1}(z)=\Psi_x(z-2\omega_1)$, i.e.

$$\Psi_x(z)=\Psi_x(z-2\omega_1)S_x(z)\,, \tag{1.26}$$

has the following monodromy properties

$$S_x(z+h)=S_x(z),\quad S_x(z+2\omega_2)=e^{2\pi i\hat{s}/h}S_x(z)e^{-2\pi i\hat{s}/h}, \tag{1.27}$$

and can be seen as a section of a bundle on the elliptic curve with periods $(h, 2\omega_2)$.

The correspondence $A(z) \to S_x(z)$ is a direct monodromy map in the elliptic case. As in the rational case, a single-valued branches of the inverse monodromy map are defined on subspaces of coefficients $A(z)$ with fixed determinant. Isomonodromy transformations which change the positions of poles and zeros of A are constructed in a way similar to the rational case. We also construct a new type of isomonodromy transformations which *change the periods* of elliptic curves. These transformations have the form

$$A'(z) = \mathcal{R}(z+h)A(z)\mathcal{R}^{-1}(z)\,, \tag{1.28}$$

where $\mathcal{R}$ is a meromorphic solution of the difference equation

$$\mathcal{R}(z+2\omega_1+h)A(z) = \mathcal{R}(z), \tag{1.29}$$

which has the following monodromy property

$$\mathcal{R}(z+2\omega_2) = e^{2\pi i\hat{q}'/(h+2\omega_1)}\mathcal{R}(z)e^{-\pi i\hat{q}/\omega_1} \tag{1.30}$$

The existence of such transformations shows that in the elliptic case there is a certain symmetry between the periods $2\omega_\alpha$ of an elliptic curve and the step h of the difference equation. Note, that this type of symmetry for q-analog of the elliptic Bernard-Knizhnik-Zamolodchikov equations was found in [16].

2 Meromorphic solutions of difference equations and Riemann-Hilbert problem

The matrix differential equation $\partial_z\Psi = A(z)\Psi$ with rational coefficients has multi-valued holomorphic solutions on $C\backslash\{z_m\}$, where z_m are the poles of $A(z)$. The initial condition $\Psi(z_0) = 1,\ z_0 \neq z_m$, uniquely defines Ψ in the neighborhood of z_0. This simple but fundamental fact is a starting point of the analytical theory of differential equations with rational coefficients. Analytic continuation of Ψ along paths in $C\backslash\{z_m\}$ defines the monodromy representation $\mu : \pi_1(C\backslash\{z_m\}) \longmapsto GL_r$.

A construction of meromorphic solutions of the difference equations is less obvious. It can be easily reduced to a solution of the following auxiliary Riemann-Hilbert type problem.

Problem I: *To find in the strip* $\Pi_x :\ x \leq \Re\, z \leq x+1$, *a continuous matrix function* $\Phi(z)$ *which is meromorphic inside* Π_x, *and such that its boundary values on the two sides of the strip satisfy the equation*

$$\Phi^+(\xi+1) = A(\xi)\Phi^-(\xi), \quad \xi = x+iy\,. \tag{2.1}$$

If Φ is a solution of this problem, then equation (1.1) can be used to extend it to a function Ψ on the whole complex plane. *A'priori* Ψ is meromorphic outside the lines $\Re\, z = x+l,\ l \in Z$. On these lines Ψ is continuous due to (2.1). Recall a well-known property of analytic

functions: if f is a continuous function in a domain D of the plane and is holomorphic in the complement $D\backslash L$ of a smooth arc L, then f is holomorphic in D. Therefore, Ψ is meromorphic on the whole complex plane and can be regarded as a meromorphic solution of (1.1).

The function $t = \tan(\pi z)$ defines a one-to-one conformal map of the interior of Π_x onto the complex plane of the variable t with a cut between the punctures $t = \pm 1$. Under this map the problem (2.1) gets transformed to the standard Riemann-Hilbert factorization problem on the cut. Fundamental results of the theory of singular integral equations imply that the problem (2.1) always has solutions. Moreover, if the index (1.15) of the corresponding systems of singular integral equations equals zero, then for a generic $A(z)$ this problem has *sectionally holomorphic* non-degenerate solution. The later means that there exists a constant $\alpha < 1$ such that $(t \pm 1)^\alpha \Phi(t)$ is bounded at the edges of the cut. In terms of the variable z a sectionally holomorphic solution Φ_x of the Problem I is a non-degenerate holomorphic matrix function inside Π_x and such that

$$\exists \;\; 0 \le \alpha < 1, \;\; |\Phi(z)| < e^{2\pi\alpha|\Im z|}, \;\; |\Im z| \to \infty. \tag{2.2}$$

This solution is unique up to the transformation $\Phi'(z) = \Phi(z)g, \; g \in SL_r$.

Almost all the results of this section do not require any additional information. Let us provide some details needed for asymptotic description of Ψ_x.

2.1 Regular singular equations

We begin with the case of regular singular difference equation, i.e. equation (1.1) with the coefficient $A(z)$ of the form

$$A = 1 + \sum_{m=1}^{n} \frac{A_m}{z - z_m}\,. \tag{2.3}$$

Equation (1.1) is invariant under the gauge transformations $A' = gAg^{-1}, \Psi' = g\Psi, \; g \in SL_r$. Thus, if the residue of Adz at the infinity is diagonalizable, we may assume without loss of generality that

$$K = \operatorname{res}_\infty Adz = \sum_{m=1}^{n} A_m = \operatorname{diag}(k_1, \ldots, k_r). \tag{2.4}$$

If $k_i - k_j \notin Z$, then equation (1.1) has a unique formal solution of the form

$$Y = \left(1 + \sum_{s=1}^{\infty} \chi_s z^{-s}\right) z^K\,. \tag{2.5}$$

The coefficients χ_s are defined by equations, which are obtained by substitution of (2.5) into (1.1). These equations express $[K, \chi_s] + s\chi_s$ in terms of A_i and $\chi_1, \ldots, \chi_{s-1}$, and can be recurrently solved for χ_s.

Let $\mathcal{P}_x$ be the space of continuous functions $\Phi(z)$ in the strip Π_x, which are holomorphic inside the strip and have at most polynomial growth at the infinity, i.e.

$$\Phi \in \mathcal{P}_x : \; \exists\, N, \;\; |\,\Phi\,| < |\,z\,|^N, \;\; z \in \Pi_x. \tag{2.6}$$

Lemma 2.1 *Let x be a real number such that $x \neq \Re z_j$. Then:*

(a) for $|x| >> 0$ there exists a unique up to normalization non-degenerate solution $\Phi_x \in \mathcal{P}_x$ of the Riemann-Hilbert problem (2.1);

(b) for a generic $A(z)$ the solution $\Phi_x \in \mathcal{P}_x$ exists and is unique up to normalization for all x such that $ind_x A = 0$;

(c) at the two infinities of the strip the function Φ_x asymptotically equals

$$\Phi_x(z) = Y(z) g_x^{\pm}, \quad \Im z \to \pm\infty. \tag{2.7}$$

Remark. Part (c) of the lemma means that, if $Y_{m'} = \left(1 + \sum_{s=1}^{m'} \chi_s z^{-s}\right) z^K$ is the partial sum of (2.5) then

$$\left| \Phi_x \left(Y_{m'} g_x^{\pm}\right)^{-1} - 1 \right| \leq O(|z|^{-m'-1}), \ \Im z \to \pm\infty \tag{2.8}$$

and the estimate (2.8) is uniform in the domain $z \in \Pi_{x,\varepsilon} : \ x + \varepsilon \leq \Re z \leq x + 1 - \varepsilon$ for any $\varepsilon > 0$.

Proof. First let us show that if Φ_x exists, then it is unique up to the normalization. The determinant of Φ_x is a holomorphic function inside Π_x. Its boundary values on the two sides of the strip satisfy the relation: $\ln\det \Phi_x^+(\xi + 1) = \ln\det \Phi_x^-(\xi) + \ln\det A(\xi)$. If $ind_x A = 0$, then the principal part of the integral of $(d \ln\det \Phi_x)$ along the boundary of Π_x equals zero. Therefore, if Φ_x is non-degenerate at least at one point then it is non-degenerate at all the points of Π_x. Now, suppose that there are two solutions of the factorization problem, then $g = \Phi_x^{-1} \Phi_x'$ is an entire periodic function. It can be regarded as a function $g(w)$ of the variable $w = e^{2\pi i z}$, holomorphic outside the points $w = 0, \ w = \infty$. From (2.6) it follows that

$$\lim_{w \to 0} w g(w) = 0, \ \lim_{w \to \infty} w^{-1} g(w) = 0. \tag{2.9}$$

Therefore, $g(w)$ has an extension which is holomorphic at the points $w = 0$ and $w = \infty$. Thus it is a constant matrix.

In a standard way the problem (2.1) is reduced to the system of linear singular equations. Let us fix for each positive integer m a holomorphic in Π_x function Y_m, which at $\pm i\infty$ coincides with Y up to the order m. If $0 \notin \Pi_x$, then we can define Y_m by the m-th partial sum of (2.5). If $0 \in \Pi_x$, then we choose $x_0 \notin \Pi_x$ and take Y_m in the form

$$Y_m = \left(1 + \sum_{s=1}^{m} \widetilde{\chi}_s (z - x_0)^{-s}\right) (z - x_0)^K, \tag{2.10}$$

where the coefficients $\widetilde{\chi}_s$ are uniquely defined by the congruence

$$\left(1 + \sum_{s=1}^{m} \widetilde{\chi}_s (z - x_0)^{-s}\right) \left(\frac{z - x_0}{z}\right)^K \left(1 + \sum_{s=1}^{m} \chi_s z^{-s}\right)^{-1} = 1 + O(z^{-m-1}). \tag{2.11}$$

Each sectionally holomorphic in Π_x function can be represented by the Cauchy type integral. Let us consider a function Φ_x given by the formula

$$\Phi_x = Y_m \phi, \quad \phi = 1 + \int_L \varphi(\xi) k(z, \xi)\, d\xi, \tag{2.12}$$

where L is the line $\Re\,\xi = x$ and

$$k(z,\xi) = \frac{e^{\pi i(z-x)} + e^{-\pi i(z-x)}}{(e^{\pi i(\xi-x)} + e^{-\pi i(\xi-x)})(e^{\pi i(\xi-z)} - e^{-\pi i(\xi-z)})}\,. \tag{2.13}$$

Let H be the space of Hëlder class functions on L, such that

$$\varphi \in H : \exists\, \alpha < 1,\ |\,\varphi(\xi)| < O(e^{\pi\alpha|\Im\,\xi|}). \tag{2.14}$$

If $\varphi \in H$, then the integral in (2.12) converges and defines a function ϕ, which is holomorphic inside Π_x, and is continuous up to the boundary. The boundary values $\phi^{\pm}$ of ϕ are given by the Sokhotski-Plemelj formulae

$$\phi^{-}(\xi) = 1 + I_\varphi(\xi) - \frac{\varphi(\xi)}{2}, \quad \phi^{+}(\xi+1) = 1 + I_\varphi(\xi) + \frac{\varphi(\xi)}{2}, \tag{2.15}$$

where $I_\varphi(\xi)$ denotes the principle value of the integral

$$I_\varphi(\xi) = p.v. \int_L \varphi(\xi')k(\xi,\xi')\,d\xi'\,. \tag{2.16}$$

The equation (2.1) is equivalent to the following nonhomogeneous singular integral equation

$$(\widetilde{A}+1)\,\varphi - 2(\widetilde{A}-1)I_\varphi = 2(\widetilde{A}-1). \tag{2.17}$$

where

$$\widetilde{A} = Y_m(\xi+1)^{-1}A(\xi)Y_m(\xi). \tag{2.18}$$

By definition of Y_m, for large $|\,z|$ we have

$$|\widetilde{A}(\xi) - 1| \le O\left(|\xi|^{-m+\kappa}\right), \quad \kappa = \max_{ij}|k_i - k_j|. \tag{2.19}$$

For large $|x|$ the left hand side of (2.19) is uniformly bounded by $O\left(|x|^{-m+\kappa}\right)$, and equation (2.17) can be solved by iterations.

Consider a sequence of the functions φ_n defined recurrently by the equation

$$(\widetilde{A}+1)\,\varphi_n - 2(\widetilde{A}-1)I_{\varphi_{n-1}} = 2(\widetilde{A}-1), \tag{2.20}$$

where $\varphi_0 = 0$. For $n > 0$ equation (2.20) implies

$$(\widetilde{A}+1)\,(\varphi_{n+1} - \varphi_n) = 2(\widetilde{A}-1)I_{(\varphi_n - \varphi_{n-1})}. \tag{2.21}$$

Therefore, if the norm of $(\widetilde{A}-1)$ is small enough, then $|\varphi_{n+1} - \varphi_n| < c\,\varepsilon^n,\ \varepsilon < 1$. The sequence φ_n obviously converges to a continuous function φ, which is a solution of (2.17). Moreover, by standard arguments used in the theory of boundary value problems (see [15] for details) it can be shown that φ is a Hëlder class function, and thus the first statement of the Lemma is proven.

For any x the left hand side of (2.17) is a singular integral operator $K : H \longmapsto H$. It has the Fredholm regularization. Furthermore, the fundamental results of the theory of

the Fredholm equations imply that nonhomogeneous linear equation (2.17) is solvable if the adjoint homogeneous equation

$$f(\xi)(\widetilde{A}(\xi)+1) - 2\left(p.v. \int_L f(\xi')k(\xi',\xi)d\xi'\right)(\widetilde{A}(\xi)-1) = 0, \tag{2.22}$$

for a (row) vector-function $f \in H_0$ has no solutions (see §53 [15]). Here H_0 is the space of the Hëlder class functions that are *integrable* on L. Each solution of (2.22) defines the (row) vector-function

$$F(z) = \cos^2(\pi(z-x))\left(\int_L f(\xi)k(\xi,z)\,d\xi\right)Y_m^{-1}(z), \tag{2.23}$$

which is a solution of the dual factorization problem in Π_x

$$F(\xi+1)A(\xi) = F(\xi), \quad \xi \in L. \tag{2.24}$$

The Cauchy kernel $k(\xi, z)$ has a simple pole at $x' = x + 1/2$. Therefore, F is holomorphic inside Π_x and equals zero at x', $F(x') = 0$. It is bounded as $|\Im z| \to \infty$. Non-existence of such solution F is an open condition. That implies the second statement of the lemma.

From (2.17, 2.19) it follows that I_φ is bounded at the infinity and $|\varphi(\xi)| < O(|\xi|^{-m+\kappa})$. Let us show that for $z \in \Pi_{x,\varepsilon}$

$$\phi(z) = g^{\pm} + O(|z|^{-m+\kappa+1}), \ \Im z \to \pm\infty, \tag{2.25}$$

where

$$g^{\pm} = 1 - \frac{1}{2}\int_L (\tan(\pi i(\xi-x)) \pm 1)\,\varphi(\xi)\,d\xi. \tag{2.26}$$

Consider the case $\Im z \to \infty$. The integral in (2.12) can be represented as the sum of two integrals I_1 and I_2. The first one is taken over the interval $L_1 : (x - i\infty, \xi_0)$ and the second one over the interval $L_2 : (\xi_0, x + i\infty)$, where $\xi_0 = x + i\Im z/2$. In $\Pi_{x,\varepsilon}$ the Cauchy kernel is uniformly bounded $k(z, \xi) < C$. Therefore,

$$|I_2| < C\int_{L_2} |\varphi(\xi)|\,d\xi < O(|z|^{-m+\kappa+1}). \tag{2.27}$$

For $\xi \in L_1$ we have $|\xi - z| > \Im z/2$. Therefore,

$$k(z,\xi) = k_+(\xi)(1 + O(e^{-\pi|z|})), \ k_+(\xi) = (1 - \tan(\pi i(\xi - x))\,, \ \xi \in L_1 \tag{2.28}$$

Hence,

$$\left|I_2 + 1 - g^+\right| < \left|\int_{L_2} k_+(\xi)\,\varphi(\xi)d\xi\right| + O(e^{-\pi|z|})\int_L k_+(\xi)\,|\varphi(\xi)|d\xi < O(|z|^{-m+\kappa+1}) \tag{2.29}$$

The proof of (2.25) for $\Im z \to -\infty$ is identical.

The solution of the factorization problem is unique. Therefore, the left hand side of (2.12) does not depend on m. Equation (2.25) implies (2.8) for $m' < m - 2\kappa$. Now letting $m \to \infty$ we obtain that (2.8) is valid for any m' and the proof of the lemma is completed.

Theorem 2.1 *If $A_0 = 1$ and $k_i - k_j \notin Z$, then:*

(A) there are unique meromorphic solutions Ψ_l and Ψ_r of equation (1.1) which are non-degenerate, holomorphic, and asymptotically represented by $Y(z)$ in the domains $\Re z << 0$ and $\Re z >> 0$, respectively [1]*;*

(B) the matrix $S = \Psi_l^{-1}\Psi_r$ has the form

$$S(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \quad S_\infty = 1 + \sum_{m=1}^{n} S_m = e^{2\pi i K}, \tag{2.30}$$

The first statement and the form of the connection matrix $S(z)$ are known (see Theorem 10.8 in [13]). The author has not found in literature an explicit form of the matrix S_∞. Birkhoff proved that $S_\infty = 1$ for the regular equations, where $K = 0$. In [13] it is stated only that S_∞ is non-degenerate.

Proof. The function Φ_x, when it exists, defines a meromorphic solution Ψ_x of the difference equation (1.1).

Lemma 2.2 *Let $x < y$ be real numbers such that the corresponding boundary problems (2.1) have solutions in $\mathcal{P}_x$ and $\mathcal{P}_y$, respectively. Then function $M_{x,y} = \Psi_y^{-1}\Psi_x$ has the form*

$$M_{x,y} = 1 - \sum_{k\in\, J_{x,y}} \frac{m_{k,(x,y)}}{e^{2\pi i(z-z_k)} - 1}, \tag{2.31}$$

where the sum is taking over a subset of indices $J_{x,y}$ corresponding to the poles such that $x < \Re z_k < y$.

Proof. By definition Ψ_x is holomorphic in Π_x. In the domain $\Re z > x + 1$ it has poles at the points $z_k + l,\ l = 1, 2 \ldots$, for $\Re\, z_k > x$. Therefore, the function $M_{x,y}$ in Π_y has poles at the points congruent to $z_k,\ k \in\ J_{xy}$. The function $M_{x,y}$ is a periodic function of z. The same arguments, as ones used above for the proof of the uniqueness of Φ_x, show that $M_{x,y}(w)$ considered as a function of the variable $w = e^{2\pi i z}$ has holomorphic extension to the points $w = 0,\ w = \infty$. Hence $M_{x,y}(w)$ is a rational function of the variable w. It equals 1 at $w = 0$ and has poles at the points $w_k = e^{2\pi i z_k},\ k \in\ J_{xy}$. Therefore, $M_{x,y}$ has the form (2.31).

Remark. The proof of the Lemma shows also that the existence of Φ_x for a generic A and x such that $ind_x A = 0$ is a simple direct corollary of the existence of Ψ_l. Indeed, let M_x be a function of the form (2.31), where the sum is taken over all $z_k :\ \Re z_k < x$. Then the condition that the function $\Psi_x = \Psi_l M_x^{-1}$, is holomorphic in Π_x is equivalent to a system of algebraic equations on the residues of M_x. If $ind_x A = 0$, then the number of equations is equal to the number of unknowns. Therefore, for a generic A the canonical meromorphic solution Ψ_x of (1.1) does exist.

The Lemma implies that Ψ_x is locally x-independent. In particular, Ψ_x is x-independent in the infinite interval $x < \min_k\{\Re z_k\}$. The corresponding function Ψ_l is the unique meromorphic solution of equation (1.1), which is holomorphic at $\Re\, z << 0$ and asymptotically

[1] In the asymptotic equalities $\Psi_{r(l)} = Y$ we assume the choice of the single-valued branch of $\ln z$ on C with a cut $arg\, z = \pi/2$

represented by Y, when $\Im z \to -\infty$, and asymptotically represented by Yg_l, when $\Im z \to \infty$. For large $|x|$ the coefficient $(\widetilde{A} - 1)$ of equation (2.17) is uniformly bounded. Therefore, φ, which decays as $|\varphi(\xi)| < O(|\xi|^{-m+k})$ at the two edges of L is also uniformly bounded by $O(|x|^{-m+k})$. Then from equation (2.26) it follows that $g_x^{\pm} = 1 + O(|x|^{-m+k})$. The matrix $g_l = g_x^+ \, (g_x^-)^{-1}$ is x-independent. Hence, $g_l = 1$ and Ψ_l is asymptotically represented by Y in the whole half-plane $\Re z << 0$. The same arguments show that Ψ_x for $x >> 0$ can be identified with Ψ_r. The statement (A) of the theorem is proved.

The formula (2.30) is a particular case of the formula (2.31). In order to complete the proof of the statement (B), we recall that the definition of Y, and therefore, the normalization of Ψ_x requires to fix a branch of $\ln z$. In our consideration it was always fixed on the z plane with a cut along the positive half of the imaginary axis. In this case, the evaluation of S at $-i\infty$ equals 1, and its evaluation at $i\infty$ is equal to the ratio of z^K on two edges of the cut.

2.2 Local monodromies.

The necessary condition for the existence of a solution Φ_x of the boundary value problem (2.1) is the equation $ind_x A = 0$. If this condition is satisfied for all the values of x then we define a notion of local monodromies of difference equations (1.1).

Special regular singular equations. We call regular singular equation (1.1) *special*, if the residues A_i of $A(z)$ are rank 1 matrices

$$A(z) = 1 + \sum_{k=1}^{n} \frac{p_k q_k^T}{z - z_k} \tag{2.32}$$

and the determinant of A identically equals 1, $\det A(z) = 1$. Here p_k, q_k are r-dimensional vectors, considered modulo transformations

$$p_k \longrightarrow c_k p_k, \quad q_k \longrightarrow c_k^{-1} q_k \,, \tag{2.33}$$

where c_k are scalars. The space of such matrices is of dimension $2N(r-2)$ and will be denoted by $\mathcal{A}_0$. Explicit parameterization of an open set of the space $\mathcal{A}_0$ can be obtained, if we order the poles, and represent $A(z)$ in the multiplicative form

$$A(z) \in \mathcal{A}_0 : \; A(z) = \left(1 + \frac{a_n b_n^T}{z - z_n}\right) \cdots \left(1 + \frac{a_1 b_1^T}{z - z_1}\right) , \tag{2.34}$$

where a_k, b_k are pairs of orthogonal vectors

$$b_k^T a_k = 0 \,, \tag{2.35}$$

considered modulo the transformation (2.33). Equation (2.35) implies

$$\left(1 + \frac{a_k b_k^T}{z - z_k}\right)^{-1} = \left(1 - \frac{a_k b_k^T}{z - z_k}\right) \longmapsto \det \left(1 + \frac{a_k b_k^T}{z - z_k}\right) = 1. \tag{2.36}$$

From (2.34, 2.35) it follows that the parameters p_k, q_k in the additive representation (2.32) of A satisfy the constraints

$$q_k^T l_k^{-1} p_k = 0, \quad l_k = 1 + \sum_{m\neq k}^{N} \frac{p_m q_m^T}{z_k - z_m}\,. \tag{2.37}$$

For matrices $A \in \mathcal{A}_0$ the gauge fixing assumption (2.4) has the form

$$\sum_{m=1}^{n} p_m q_m^T = \sum_{m=1}^{n} a_m b_m^T = \operatorname{diag}(k_1, \ldots, k_r) = K\,. \tag{2.38}$$

It is assumed throughout this subsection that the real parts of the poles $r_k = \Re z_k$ are distinct, and $r_k < r_m,\ k < m$. For further use, we introduce also the notation $r_0 = -\infty, r_{n+1} = \infty$.

Theorem 2.2 (i) *For a generic matrix $A \in \mathcal{A}_0$, satisfying (2.38), where $k_i - k_j \notin Z$, the corresponding special regular singular equation (1.1) has a set of unique meromorphic solutions $\Psi_k,\ k = 0, \ldots, n$, which are holomorphic in the strips $r_k < \Re z < r_{k+1} + 1$ and asymptotically represented by $Y g_k^{\pm}$, as $\Im z \to \pm\infty$, where $g_k^- = 1$.*

(ii) *The local connection matrices $M_k = \Psi_k^{-1}\Psi_{k-1},\ k = 1, \ldots, n$, have the form*

$$M_k = 1 - \frac{\alpha_k \beta_k^T}{e^{2\pi i(z-z_k)} - 1} \tag{2.39}$$

where (α_k, β_k) are pairs of orthogonal vectors

$$\beta_k^T \alpha_k = 0, \tag{2.40}$$

considered modulo transformations (2.33) and such that

$$(1 + \alpha_n \beta_n^T) \cdots (1 + \alpha_1 \beta_1^T) = e^{2\pi i K} \tag{2.41}$$

(iii) *The map of pairs of orthogonal vectors $\{a_m, b_m\} \longmapsto \{\alpha_k, \beta_k\}$, considered modulo transformation (2.33), is a one-to-one correspondence of open sets of the varieties defined by the constraints (2.35, 2.38) and (2.40, 2.41), respectively.*

Proof. As it was shown above, a solution $\Phi_x \in \mathcal{P}_x$ of the factorization problem (2.1) exists if the homogeneous singular integral equation (2.22) has no solutions. That is an open type condition and therefore for generic A the corresponding meromorphic solution Ψ_x of equation (1.1) does exist. If $r_k < x < r_{k+1}$, then equation (1.1) implies that Ψ_x has poles at the points $z_m + l,\ m = 1, 2 \ldots$, for $k < m$ and at the points $z_m - l,\ l = 0, 1, \ldots$, for $m \le k$. Therefore, Ψ_x is holomorphic in the strip $r_k < \Re z < r_{k+1} + 1$ and can be identified with Ψ_k. The solutions Ψ_k exists for all k, if A belongs to the intersection of open sets corresponding to each k. It is still an open condition, therefore, Ψ_k do exist for a generic A. They are unique and have asymptotic representation, described in (i).

The residues of $A(z)$ are rank 1 matrices. Therefore, the residue of M_k at z_k is also a rank 1 matrix and can be represented in the form $\alpha_k \beta_k^T$, where α_k, β_k are vectors defined

up to the transformation (2.33). Then (2.31) implies equation (2.39). From the constraint $\det A = 1$ and the normalization $g_k^- = 1$ it follows that $\det \Psi_k = 1$. Hence, $\det M_k = 1$. That implies (2.40). The global connection matrix is the product of local ones, $S = M_n \cdots M_1$. Therefore, equation (2.30) implies (2.41) and thus the second statement of the Theorem is proven.

Now let us show that the map $\{a_m, b_m\} \longmapsto \{\alpha_k, \beta_k\}$ is injective on the open set of matrices $A \in \mathcal{A}_0$ for which the corresponding difference equation has a set of canonical solutions Ψ_k. Indeed, suppose that there exist two special regular singular equations having the same local connection matrices. Then we have two sets of the corresponding meromorphic solutions Ψ_k and Ψ_k' which are holomorphic in the strips $\Re z \in (r_k, r_{k+1}+1)$, and which are asymptotically equal to $O(1)z^K g_k^{\pm}$, $g_k^- = 1$ as $\Im z \to \pm\infty$. Note, that the matrices g_k^+ are the same for Ψ_k and Ψ_k' because they are equal to the products of *the monodromy matrices* $\mu_k = 1 + \alpha_k \beta_k^T$

$$g_0^+ = 1, \quad g_k^+ = \mu_{k-1} \cdots, \mu_1, \; k > 1 \,. \tag{2.42}$$

The matrix function, which equals $\Psi_k' \Psi_k^{-1}$ in each of the corresponding strips is continuous across the boundaries. Hence, it is an entire function which is bounded at the infinity. It tends to 1 as $\Im z \to -\infty$. Therefore, it equals 1 identically.

The proof of a surjectivity of the map $\{a_m, b_m\} \longmapsto \{\alpha_k, \beta_k\}$ on an open set of the connection matrices once again is reduced to the Riemann-Hilber type factorization problem. Let us fix a small enough real number ε. Then, the vertical lines $L_m : \Re \xi = \Re z_m + \varepsilon$ divide the complex plane into $(n+1)$ domains $\mathcal{D}_k$, $k = 0, \ldots, n$.

Problem II: *For a given set of matrix functions $\mathcal{M}_j(\xi)$ on L_j find matrix functions $\mathcal{X}_k(z)$, which are holomorphic inside the domains $\mathcal{D}_k$, continuous up to the boundaries, and whose boundary values satisfy the equation*

$$\mathcal{X}_{k-1}^+(\xi) = \mathcal{X}_k^-(\xi) \mathcal{M}_k(\xi), \quad \xi \in L_k \,. \tag{2.43}$$

Let M_k be a set of matrices of the form (2.39) satisfying the constrains (2.40,2.41). Then we consider first the Problem II for the set of piece-wise constant matrices

$$\mathcal{M}_k^0(\xi) = 1, \; \Im \xi \geq 0, \; \mathcal{M}_k^0(\xi) = \mu_k, \; \Im \xi < 0. \tag{2.44}$$

This is just the inverse monodromy problem for differential equation, solved by Plemelj. He showed that the solution of this problem exists if at least one of the monodromy matrices is diagonalizable [17]. Let $\mathcal{F}_k$ be a solution of this auxiliary problem. Then we define a new set of functions $\mathcal{M}_k(\xi)$ by the formula

$$\mathcal{M}_k = \mathcal{F}_k^+ M_k \left(\mathcal{F}_k^-\right)^{-1} . \tag{2.45}$$

The function M_k tends to μ_k exponentially, as $\Im z \to \infty$. Therefore, $\mathcal{M}_k \to 1$ at both the edges of L_k. In that case we may find a solution of the problem (2.43) in the form of the Cauchy integral

$$\mathcal{X}(z) = 1 + \sum_k \int_{L_k} \frac{\chi_k(\xi) d\xi}{\xi - z} \,. \tag{2.46}$$

Inside each of the domains $\mathcal{D}_k$ formula (2.46) defines a holomorphic function $\mathcal{X}_k$. Using the Sokhotski-Plemelj formulae for their boundary values we obtain the system of singular integral equations for χ_k

$$\frac{1}{2}\chi_k(\xi)(\mathcal{M}_k(\xi)+1) - \frac{1}{2\pi i}I_\chi(\xi)(\mathcal{M}_k(\xi)-1) = (\mathcal{M}_k(\xi)-1), \tag{2.47}$$

where $I_\chi(\xi)$ denotes the principle value of the integral

$$I_\chi(\xi) = p.v. \sum_k \int_{L_k} \frac{\chi_k(\xi')d\xi'}{\xi'-\xi}\,. \tag{2.48}$$

The non-homogeneous term of the system tends to zero at the infinity. Therefore, for a generic set of matrices M_k the system has a solution in the space of Hëlder class functions decaying at infinity. That implies that $\mathcal{X}_k$ tends to the identity matrix at the infinity. The functions $\mathcal{F}_k$ have asymptotic behavior $O(1)z^K g_k^\pm$. Hence the functions $\Psi_k = \mathcal{X}_k\mathcal{F}_k$ have the same asymptotic behavior. Its boundary values satisfy the relation

$$\Psi_{k-1}^+(\xi) = \Psi_k^-(\xi)M_k(\xi), \quad \xi \in L_k\,. \tag{2.49}$$

This equation can be used for the meromorphic extension of Ψ_k on the whole complex plane. At the same time it shows that the function $A_k(z) = \Psi_k(z+1)\Psi_k^{-1}(z)$ is k-independent. In the domain $\mathcal{D}_k$ it has a unique simple pole at z_k. Therefore, $A(z)$ is a meromorphic function with simple rank 1 poles at the points z_k. It tends to the identity matrix at the infinity and $\det A = 1$, i.e. $A \in \mathcal{A}_0$ and thus the Theorem is proven.

Unitary difference equations. As it has been emphasized above, for a given real number x the canonical meromorphic solution Ψ_x exists only for generic difference equations. Here is an example of the class of difference equations for which the canonical solutions *always exist.*

We call the difference equation unitary, if its coefficient satisfies the condition

$$A(z) \in \mathcal{A}^U: \quad A^+(\bar{z}) = A^{-1}(z)\,, \tag{2.50}$$

where A^+ is the hermitian conjugate of A. An open set of such matrices can be parameterized by the sets of unit vectors a_k

$$A(z) = \prod_{k=1}^{n}\left(1 + a_k a_k^+ \frac{z_k - \bar{z}_k}{z - z_k}\right), \quad a_k^+ a_k = |a_k|^2 = 1. \tag{2.51}$$

The factors in the product (2.51) are ordered so that the indices increase from right to left. Recall that in this section we assume that the residue of A at the infinity is a diagonal matrix

$$\sum_{k=1}^{n}(z_k - \bar{z}_k)a_k a_k^+ = K, \quad K^{ij} = k_i\delta^{ij}, \quad k_i - k_j \notin Z. \tag{2.52}$$

Equation (2.50) implies $\det \bar{A}(\bar{z}) = \det A^{-1}(z)$. Therefore, for any $x \neq \Re z_k$ the index of the boundary problem (2.1) equals zero, $ind_x\, A = 0$.

Lemma 2.3 *Let $A(z)$ be the coefficient of a regular singular unitary equation. Then for each $x \neq \Re z_k$ the boundary problem (2.1) has non-degenerate solution $\widetilde{\Phi}_x \in \mathcal{P}_x$ such that*

$$\widetilde{\Phi}_x^+(\bar{z}) = \widetilde{\Phi}_x^{-1}(z). \tag{2.53}$$

This solution is unique up to a unitary normalization

$$\widetilde{\Phi}'_x(z) = \widetilde{\Phi}_x(z)\, u, \quad u \in U(r)\,. \tag{2.54}$$

Proof. As it was shown above, the Riemann-Hilbert problem (2.1) has a solution $\Phi \in P_x$, if the dual boundary problem (2.24) has no vector solution which is bounded at $-i\infty$ and tends to zero faster then any negative power of $\Im z$ at the other edge of the strip. Suppose that such vector solution F exists. Then the scalar function $F(z)F^+(\bar{z})$ is holomorphic in Π_x and tends to zero at both edges of the strip. Therefore, the integral of this function over the boundary of the upper half Π_x^+ of the strip Π_x exists and equals zero,

$$\oint_{\partial\Pi_x^+} F(z)F^+(\bar{z})dz = 0, \quad z \in \Pi_x^+ \subset \Pi_x : \Im z \geq 0. \tag{2.55}$$

On the other hand, from (2.50) it follows that this function is periodic, i.e. its evaluations at $\xi = x + iy$ and $\xi + 1$ are equal. Therefore, the integral (2.55) equals the integral over the bottom edge of Π_x^+

$$\oint_{\partial\Pi_x^+} F(z)F^+(\bar{z})dz = \int_x^{x+1} |F(x')|^2 dx' > 0. \tag{2.56}$$

The contradiction of (2.55) and (2.56) implies that Φ_x exists. It was shown earlier that Φ_x is unique up to normalization. Let us normalize it by the condition that Φ_x has asymptotic Y as $\Im z \to -\infty$. At the other edge of the strip it has asymptotic $Y g_x$ (in this subsection we don't use notations $g_x^\pm$ in order to avoid confusing them with the sign of the hermitian conjugation.)

Our next goal is to show that g_x is a positively defined hermitian matrix. Indeed, from (2.50) it follows that if Φ_x is a solution of the boundary problem, then the matrix $(\Phi_x^+(\bar{z}))^{-1}$ is also a solution of the same problem. That implies

$$\left(\Phi_x^+(\bar{z})\right)^{-1} = \Phi_x(z)h, \quad h \in GL_r. \tag{2.57}$$

The evaluation of this equality at two edges of the strip gives $gh = 1$ and $g^+h = 1$. Hence, $g = g^+$. The matrix $\Phi_x^+(\bar{z})\Phi_x(z)$ is holomorphic in Π_x and has equal values on two sides of the strip. Hence, for any vector v we have

$$\oint_{\partial\Pi_x^+} v^+\, \Phi_x^+(\bar{z})\Phi_x(z)\, v dz = 0 \;\longmapsto\; v^+\, g\, v = \int_x^{x+1} v^+\, \Phi_x^+(x')\Phi_x(x')\, v dx' > 0. \tag{2.58}$$

Thus g is positively defined, and therefore there exists a matrix g_1 such that $g = g_1^+ g_1$. Equation (2.57) implies that the function $\widetilde{\Phi}_x = \Phi_x g_1^{-1}$ satisfies (2.53).

Theorem 2.3 *Let $A(z)$ be a matrix of the form (2.51). Then:*

(i) *the corresponding difference equation (1.1) has a unique set of meromorphic solutions $\widetilde{\Psi}_k$, such that: (a) $\widetilde{\Psi}_k$ is holomorphic in the strip $r_k < \Re z < r_{k+1} + 1$, and grows at most polynomially, as $\Im z \to \pm\infty$; (b) $\widetilde{\Psi}_0 = (1 + O(z^{-1}))\, z^K$, $\Im z \to -\infty$; (c) $\widetilde{\Psi}_k$ satisfies the relation*

$$\widetilde{\Psi}_k^+(\bar{z}) = \widetilde{\Psi}_k^{-1}(z)\,; \tag{2.59}$$

(d) *the local connection matrices $M_k = \widetilde{\Psi}_k^{-1}\widetilde{\Psi}_{k-1}$ have the form*

$$\widetilde{M}_k(z) = 1 - f_k(z)\alpha_k\alpha_k^+, \tag{2.60}$$

where

$$f_k(z) = (1 + |w_k|)\,\frac{ww_k^{-1} - |w_k|^{-1}}{ww_k^{-1} - 1},\ \ w = e^{2\pi iz},\ \ w_k = w(z_k) \tag{2.61}$$

and α_k are unit vectors, $\alpha_k^+\alpha_k = 1$, satisfying the constraint

$$(1 - \nu_n\alpha_n\alpha_n^+)\cdots(1 - \nu_1\alpha_1\alpha_1^+) = e^{\pi iK}, \quad \nu_k = 1 + |w_k|. \tag{2.62}$$

(ii) *The monodromy map of sets of unit vectors $\{a_k\} \longmapsto \{\alpha_k\}$ is a one-to-one correspondence of the varieties defined by equations (2.52) and (2.62).*

Proof. Lemma 2.3 implies that solutions $\widetilde{\Psi}'_k$ satisfying conditions (a) and (c) exist and are unique up to normalization. The corresponding connection matrix $\widetilde{M}'_k$, which is a rational function of w, satisfies the equation

$$\widetilde{M}_k'^{+}(\bar{z}) = \widetilde{M}_k'^{-1}(z) \tag{2.63}$$

and has the only pole at w_k, where its residue is a rank 1 matrix. It is easy to check that each matrix, which satisfies these properties has a unique representation in the form $\widetilde{M}'_k = u_k\widetilde{M}_k$, where $\widetilde{M}_k$ is given by (2.60) and $u_k \in U(r)$. The condition (b) uniquely normalizes $\widetilde{\Psi}_0$. Then, under the change of the normalization $\widetilde{\Psi}'_k = \widetilde{\Psi}_k u_k$, $\ u_k \in U(r)$, the local connection matrices get transformed to $\widetilde{M}_k$.

The global connection matrix $\widetilde{S} = \widetilde{M}_n \cdots \widetilde{M}_1$ up to a z-independent factor is equal to the global connection matrix S corresponding to the canonically normalized solutions Ψ_k used before, i.e. $\widetilde{S} = \widetilde{S}(-i\infty)S(z)$. Therefore, using (2.63) we get $S(i\infty) = \widetilde{S}^{-1}(-i\infty)\widetilde{S}(i\infty) = \widetilde{S}^2(i\infty)$. The left hand side of (2.62) equals $\widetilde{S}(i\infty)$. Therefore, equation (2.30) implies (2.62).

The proof of (ii) is almost identical to that of the last statement of Theorem 2.2.

Small norm case. Now we are in the position to present another case, for which once again the notion of monodromies around the poles of $A(z)$ can be introduced. This case is of special importance for further considerations.

For simplicity, it is assumed throughout this subsection that $\Re z_k < \Re z_m,\ k < m$. Let us fix a number $\varepsilon << \max_{km}|\Re z_k - \Re z_m|$ and consider the space of matrix functions $A(z)$ of the form (2.3) such that the euclidian norm $|A_k| < \varepsilon/2$. If ε is small enough, then $A(z)$

is invertible for $|z-z_k|>\varepsilon$, and therefore, zeros of $\det A$ are localized in the neighborhoods of the poles. Let us denote them by z_{ks}^-:

$$\det A(z_{ks}^-)=0, \quad |z_k-z_{ks}^-|<\varepsilon, \quad s=1,\dots,\; h_k=\operatorname{rank} A_k. \tag{2.64}$$

Furthermore, for small enough ε a solution of the singular equation (2.17) for $x_k=(\Re z_k+\Re z_{k+1})/2,\; k=1,\dots,n-1$, can be constructed by the same iterations (2.20) as it was done before for $|x|>>0$. The corresponding canonical solution $\Psi_k=\Psi_{x_k}$ of (1.1) has poles at the points $z_m+l,\; l=1,\dots,\; k\leq m$ and at the points $z_{ms}^- -l,\; l=0,\dots,\; \; m\leq k$. Then, along lines identical to those used for the proof of Theorem 2.2, we obtain the following statement.

Theorem 2.4 *There exists ε such that, if $|A_k|<\varepsilon$ and satisfy (2.4), then the corresponding regular singular equation (1.1) has a set of unique meromorphic solutions $\Psi_k,\; k=0,\dots,n$, which are holomorphic in the strips $r_k+\varepsilon<\Re z<r_{k+1}+1$, and grow at most polynomially as $|\Im z|\to\infty$, and are normalized by the condition: $\lim_{\Im z\to-\infty}\Psi_k z^{-K}=1$.*

(i) The solutions Ψ_k are asymptotically represented by $Yg_k^{\pm}$, as $\Im z\to\pm\infty$; $g_k^-=1$, $g_0^+=g_n^+=1$.

(ii) The local connection matrices $M_k=\Psi_k^{-1}\Psi_{k-1},\; k=1,\dots,n$, have the form

$$M_k=1-\frac{m_k}{e^{2\pi i(z-z_k)}-1}, \tag{2.65}$$

where m_k are matrices such that

$$(1+m_n)\dots(1+m_1)=e^{2\pi iK}. \tag{2.66}$$

(iii) The map $\{A_m\}\longmapsto\{m_k\}$ is a one-to-one correspondence of the space of matrices $|A_m|<\varepsilon$, satisfying (2.4), and an open neighborhood of the point $(m_k=0)$ of the variety defined by equation (2.66).

2.3 Mild equations

In this subsection the previous results are extended to the case of mild differential equations (1.1) with diagonalizable leading coefficient

$$A=A_0+\sum_{m=1}^{n}\frac{A_m}{z-z_m}, \quad A_0^{ij}=\rho_i\delta^{ij} \tag{2.67}$$

If $\rho_i\neq\rho_j$, then (1.1) has unique formal solution of the form (1.5). The substitution of (1.5) into (1.1) gives a set of equations for χ_s. The first nontrivial equation

$$[A_0,\chi_1]=\sum_{m=1}^{n}A_m-K \tag{2.68}$$

defines the diagonal matrix

$$K^{ij}=k_i\delta^{ij}, \quad k_i=\sum_{m=1}^{n}A_m^{ii}, \tag{2.69}$$

and the off-diagonal part of the matrix χ_1. On each step the consecutive equation defines recurrently the diagonal entries of χ_{s-1} and the off-diagonal part of χ_s.

First, let us consider the case of the **real exponents**.

Theorem 2.5 *Let A be a matrix of the form (2.67) with $\rho_i \neq \rho_j$, $\Im\rho_i = 0$. Then:*

(A) there are unique meromorphic solutions Ψ_l, Ψ_r of equation (1.1), which are holomorphic in the domains $\Re z << 0$ and $\Re z >> 0$, respectively, and which are asymptotically represented by $Y g^{\pm}_{l(r)}$, $g^{-}_{l(r)} = 1$, as $\Im z \to \pm\infty$; the matrices $g_{r(l)} = g^{+}_{r(l)}$ satisfy the constraints (1.11);

$$g^{ii}_{r(l)} = 1, \;\; g^{ij}_r = 0, \text{ if } \rho_i < \rho_j, \quad g^{ij}_l = 0, \text{ if } \rho_i > \rho_j, \tag{2.70}$$

(B) the connection matrix $S = (\Psi_r)^{-1}\Psi_l$ has the form

$$S(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \;\; S_\infty = 1 + \sum_{m=1}^{n} S_m = g_r^{-1} e^{2\pi i K} g_l\,; \tag{2.71}$$

If the case of real exponents $\Im\rho_i = 0$ the matrix $e^{z \ln A_0 + zK}$ grows at most polynomially as $|\Im z| \to \infty$, and almost all the results proved above for the regular singular equations hold. Lemma 2.1 does not require any changes at all. As before, it implies the existence of meromorphic canonical solutions Ψ_r and Ψ_l of (1.1). These solutions are asymptotically represented by $Y g^{\pm}_{l(r)}$, $\Im z \to \pm\infty$. They are uniquely normalized by the condition $g^{-}_{l(r)} = 1$. The only difference of mild equations with distinct real exponents and regular singular equations is that for the first ones equation $g_{l(r)} = 1$ does not hold. The coefficient $(\widetilde{A} - 1)$ in (2.17) is of the form

$$\widetilde{A} - 1 = e^{-z \ln A_0 - K \ln z}\, 0(z^{-m})\, e^{z \ln A_0 + K \ln z} \tag{2.72}$$

From (2.17) it follows that φ asymptotically has the quasi-triangular form. Then equation (2.26) implies (2.70). The proof of (2.71) is identical to that of (2.30).

Let us consider now the **Birkhoff's case** of exponents ρ_i with distinct imaginary parts of $\ln\rho_i$. Below we assume that the branch of $\ln\rho_i$ is chosen such that

$$-\pi < \nu_i = \Im(\ln\rho_i) \leq \pi. \tag{2.73}$$

Theorem 2.6 *Let A be a matrix of the form (2.67) with $\nu_i \neq \nu_j \neq 0$. Then:*

(A) there are unique meromorphic solutions Ψ_l, Ψ_r of equation (1.1), which are holomorphic in the domains $\Re z << 0$ and $\Re z >> 0$, respectively, and which are asymptotically represented by Y, $\Im z \to \pm\infty$;

(B) the connection matrix $S = (\Psi_r)^{-1}\Psi_l$ has the form

$$S(z) = S_0 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \tag{2.74}$$

where S_0 and $S_\infty = 1 + \sum_{m=1}^{n} S_m$ satisfy the constraints:

$$S_0^{jj} = 1, \quad S_0^{ij} = 0, \text{ if } \nu_i > \nu_j, \quad S_\infty^{jj} = e^{2\pi i k_j}, \quad S_\infty^{ij} = 0, \text{ if } \nu_i < \nu_j. \tag{2.75}$$

The first statement of the theorem is one of the fundamental Birkhoff's results. Nevertheless, it is instructive to outline its proof via the Riemann-Hilber factorization problem (2.1). It clarifies the similarity and the difference of the Birkhoff's case and the case of real exponents. The differences are mainly due to the simple fact that in case $\nu_i \neq \nu_j$ the formal series Y and Yg are asymptotically equal to each other, as $\Im z \to \pm\infty$, if g is quasi upper- or lower-triangular matrix , respectively, whose diagonal entries equal 1. As a result, the notion of the transfer matrix g_x along the thick path Π_x introduced above has no intrinsic meaning in the Birkhoff's case. It is hidden in the normalization of $\Psi_{l(r)}$, and to some extend, re-appear in the form of the connection matrix S.

As above, the construction of a sectionally holomorphic solution Φ_x of the Riemann-Hilbert factorization problem (2.1) is reduced to a singular integral equation. Let Φ_x be a function given by the formula

$$\Phi_x = Y_m \phi, \quad \phi = g + \int_L \varphi(\xi) k(z,\xi)\, d\xi. \tag{2.76}$$

The function Φ_x is a solution of the Riemann-Hilbert problem if $\varphi \in H$ is a solution of the singular integral equation

$$(\widetilde{A} + 1)\, \varphi - 2(\widetilde{A} - 1) I_\varphi = 2(\widetilde{A} - 1) g, \tag{2.77}$$

where $\widetilde{A}$ is given by (2.18). For regular singular equations and for the case of mild equations with real exponents a choice of the constant term g in (2.76) was inessential. It becomes crucial for the case of imaginary exponents.

Our next goal is to show that there exists a unique matrix g whose diagonal entries equal $g^{ii} = 1$, and such that equation (2.77) has a solution $\varphi \in H$ with entries satisfying the conditions

$$|\varphi^{ij}(\xi)| < O(\,|y|^{-m+\kappa})\; e^{y\,\nu_{ij}}, \;\; \nu_{ij} = \nu_i - \nu_j, \quad y = \Im \xi \to \pm\infty. \tag{2.78}$$

If a smooth matrix function φ satisfies (2.78), then the corresponding Cauchy integrals have the following asymptotics

$$\pm\, \nu_{ij} > 0: \qquad \begin{cases} \left| I^{ij}_\varphi \right| & < O(\,|y|^{-m+\kappa})\; e^{y\,\nu_{ij}}, \quad y \to \pm\infty, \\ |I^{ij}_\varphi - f^{ij}_\varphi| & < O(\,|y|^{-m+\kappa})\; e^{y\,\nu_{ij}}, \quad y \to \mp\infty, \end{cases} \tag{2.79}$$

where

$$\pm\, \nu_{ij} > 0: \quad f^{ij}_\varphi = -\frac{1}{2} \int_L (\tan(\pi y) \pm 1)\, \varphi^{ij}(\xi) d\xi. \tag{2.80}$$

The proof of the second inequality in (2.79) is almost identical to that of (2.25). The first inequality can be obtained by similar arguments (see also formula () in [15]).

Self-consistence of equation (2.77) and the conditions (2.79) implies

$$g = 1 - f_\varphi\,, \tag{2.81}$$

where f_φ is an off-diagonal matrix given by (2.80). Equations (2.77) and (2.81) can be seen as a system of equations for unknown $\varphi(\xi)$ and g. This system for a large $|x|$ can be solved by iterations. For that we take $\varphi_0 = 0$ and define φ_n recurrently by the equation

$$(\widetilde{A} + 1)\, \varphi_{n+1} = 2(\widetilde{A} - 1)(1 + I_{\varphi_n} - f_{\varphi_n}), \tag{2.82}$$

From (2.79) it follows that if φ_n satisfies (2.78), then φ_{n+1} satisfies the same conditions, as well. The sequences $g_n = 1 - f_{\varphi_n}$, φ_n converge and define g and a solution φ of (2.77), which satisfies (2.79).

From (2.79) it follows that if ϕ and g are solutions of (2.77) and (2.81), then the off-diagonal entries of the matrix function Φ given by (2.76) have the asymptotic

$$|\phi^{ij}(z)| < O(\,|z|^{-m+\kappa})\ \ |(\rho_j/\rho_i)^z|\,, \quad \Im z \to \pm\infty. \tag{2.83}$$

on both the edges of $\Pi_{x,\varepsilon}$. For the diagonal elements of ϕ we have the same asymptotic as that proven above for the case of regular singular equations, i.e.

$$|\phi^{jj}(z) - v_j^{\pm}| < O(\,|z|^{-m+\kappa+1}), \quad \Im z \to \pm\infty, \tag{2.84}$$

where

$$v_j^{\pm} = 1 - \frac{1}{2}\int_L (\tan(\pi i(\xi - x)) \pm 1)\, \varphi^{jj}(\xi)\, d\xi. \tag{2.85}$$

The same arguments as used above in the section 2.1, show that, if there exists a sectionally meromorphic solution Φ_x of the Riemann-Hilbert problem (2.1), then it is unique. Therefore, (2.83,2.84) imply the following statement.

Lemma 2.4 *For a generic A, such that $\mathrm{ind}_x A = 0$ there exists a unique holomorphic solution Φ_x of the Riemann-Hilbert Problem (2.1) asymptotically represented by Y, as $\Im z \to -\infty$, and Yv_x, as $\Im z \to \infty$, where v_x is a diagonal matrix.*

For large $|x|$ the solutions Ψ_x of equation (1.1) corresponding to Φ_x are x-independent for $x >> 0$ and $x << 0$ and can be identified with the Birkhoff's solutions Ψ_r and Ψ_l, respectively. Indeed, for a large $|x|$ the functions φ^{ii} are uniformly bounded by $O(|x|^{-m+\kappa})$. Therefore, (2.84) implies $v_{l(r)} = 1$. The first statement of the theorem is proved.

From (2.73) it follows that the connection matrix S considered as a function of the variable $w = e^{2\pi i z}$ has holomorphic extension at the points $w = 0, w = \infty$. Therefore, it is a rational function of w having poles at $w_m = w(z_m)$. Hence, it has the form (2.71). Its evaluations at $w = 0$ and $w = \infty$ are quasi triangular matrices for obvious reasons. The proof of the theorem is completed.

Local monodromies for mild equations can be introduced for the same three cases considered above in the section 2.2. Namely, for the cases of *special, unitary* and *small norm* coefficients. The form of the local monodromy matrices in the case of mild equations with real exponents was described in the Introduction. Extensions of all the other results of the section 2.2 for the case of mild equations are straightforward. For example let us consider the special mild equations with imaginary exponents, satisfying the Birkhoff's condition.

Theorem 2.7 (*i*) *For a generic matrix A of the form*

$$A(z) = A_0 \left(1 + \frac{a_n b_n^T}{z - z_n}\right) \cdots \left(1 + \frac{a_1 b_1^T}{z - z_1}\right) \tag{2.86}$$

where

$$(a)\ A_0^{ij} = \rho_i \delta^{ij},\ \nu_i \neq \nu_j,\ \nu_i = \Im(\ln \rho_i), \quad (b)\ b_k^T a_k = 0, \quad (c)\ \Re z_k < \Re z_m,\ k < m,$$

equation (1.1) has a set of unique meromorphic solutions $\Psi_k,\ k = 0, \ldots, n,$ *which are holomorphic in the strips* $r_k < \Re z < r_{k+1} + 1$, *and asymptotically represented by* $Y v_k^{\pm}$, *as* $\Im z \to \pm\infty$, *where* $v_k^- = 1$, *and* v_k^+ *is a diagonal matrix*

(ii) *The local connection matrices* $M_k = \Psi_k^{-1} \Psi_{k-1},\ k = 1, \ldots, n,$ *have the form*

$$M_k = m_{k0} - \frac{\alpha_k \beta_k^T}{e^{2\pi i (z - z_k)} - 1} \tag{2.87}$$

where (α_k, β_k) *are pairs of orthogonal vectors, considered modulo transformations (2.33), and* m_{k0} *is a quasi lower-triangular matrix such that* $M_k(i\infty)$ *is a quasi upper-triangular matrix, i.e.*

$$m_{k0}^{jj} = 1, \quad m_{k0}^{ij} = 0, \text{ if } \nu_i > \nu_j, \quad m_{k0}^{ij} = -\alpha_k^i \beta_k^j, \text{ if } \nu_i < \nu_j, \tag{2.88}$$

(iii) *The map of pairs of orthogonal vectors* $\{a_m, b_m\} \longmapsto \{\alpha_k, \beta_k\}$, *considered modulo transformation (2.33), is a one-to-one correspondence of open sets.*

In the small norm case the local connection matrix M_k is described in similar terms. Namely, it has the form

$$M_k = m_{k0} - \frac{m_{k1}}{e^{2\pi i (z - z_k)} - 1}\,, \tag{2.89}$$

where m_{k0} is quasi lower-triangular matrix and $m_{k0} + m_{k1}$ is a quasi upper-triangular matrix. The discrete analog of the local monodromy matrix is defined as their ratio

$$\mu_k = 1 + m_{k1} m_{k0}^{-1} \tag{2.90}$$

Note, that a generic matrix has a unique factorization as the product of lower- and upper-triangular matrices. Therefore, equation (2.90) implies that μ_k uniquely defines the corresponding pair of matrices m_{k0}, m_{k1}, and, consequently, the local and global connection matrices.

3 The inverse monodromy problem and isomonodromy transformations

In this section we consider a map inverse to the direct monodromy map

$$\{z_m, A_m\} \longmapsto \{w_m, S_m\}, \quad w_m = w(z_m) = e^{2\pi i z_j}\,. \tag{3.1}$$

For any fixed diagonalizable matrix A_0 the characterization of equations (1.1) having the the same monodromy data is identical to that given by Birhhoff for the case of imaginary exponents.

Lemma 3.1 *Rational functions $A(z)$ and $A'(z)$ of the form (1.2) under the map (3.1) correspond to the same connection matrix $S(z)$ if and only if there exists a rational matrix function $R(z)$ such that*

$$A'(z) = R(z+1)A(z)R^{-1}(z), \quad R(\infty) = 1. \tag{3.2}$$

Proof. Let $\Psi_{l(r)}$ and $\Psi'_{l(r)}$ be canonical meromorphic solutions of equation (1.1) corresponding to $A(z)$ and $A'(z)$, respectively. If $\Psi_r^{-1}\Psi_l = (\Psi'_r)^{-1}\Psi'_l$, then

$$R = \Psi'_l \Psi_l^{-1} = \Psi'_r \Psi_r^{-1}. \tag{3.3}$$

By definition of the canonical solutions, the matrix function R is holomorphic for large $|z|$. Moreover, if $A_0 = A'_0, K = K'$, then $R \to 1, |z| \to \infty$. Hence, R has only finite number of poles, and therefore, is a rational function of the variable z.

Let $\mathcal{A}_D$ be the subspace of the space $\mathcal{A}$ of matrix functions of the form (1.2) having fixed determinant

$$A \in \mathcal{A}_D \subset \mathcal{A}: \quad \det A(z) = D(z) = \frac{\prod_{\alpha=1}^N (z - \zeta_\alpha)}{\prod_{m=1}^n (z - z_m)^{h_m}}, \quad h_m = \text{rk } A_m. \tag{3.4}$$

Note, that the constraint (1.4) is equivalent to the condition

$$\text{tr } K = 0 \longleftrightarrow \sum_\alpha \zeta_\alpha = \sum_m h_m z_m. \tag{3.5}$$

Lemma 3.2 *If the zeros ζ_α are not congruent, i.e. $\zeta_\alpha - \zeta_\beta \notin Z$, then the monodromy correspondence (3.1) restricted to $\mathcal{A}_D$ is injective.*

Proof. Let $\mathcal{A} \in \mathcal{A}_D$ be a matrix whose poles and zeros of the determinant are not congruent pairwise . Suppose that there exists a rational matrix function R that equals 1 at the infinity, i.e. $R = 1 + O(z^{-1})$, and such that the matrix A' defined by (3.3) has the same determinant, i.e. $A' \in \mathcal{A}_D$. Then the equation $R(z+1) = A'(z)R(z)A^{-1}(z)$ implies that R has poles of *constant* ranks at the points $\zeta_\alpha + l$ and $z_m + l$, where $l \in Z_+$ is a positive integer. The matrix R is regular at the infinity. Therefore, it should be regular everywhere. That implies $R = 1$.

Let us call rational functions D and D' equivalent if sets of their poles z_m, z'_m and zeros $\zeta_\alpha, \zeta'_\alpha$ are congruent to each other, i.e $z_m - z'_m \in Z, \ \ \zeta_\alpha - \zeta'_\alpha \in Z$, and satisfy the relation (3.5).

Lemma 3.3 *For each pair of equivalent rational functions D and D' there exists a unique isomonodromy birational transformation*

$$T_D^{D'} : \mathcal{A}_D \longmapsto \mathcal{A}_{D'} \tag{3.6}$$

Proof. The construction of the isomonodromy transformations $T_D^{D'}$ is analogous to that proposed in [10] for the case of polynomial coefficients $\widetilde{A}$. To begin with, we introduce two types of elementary transformations. They are birational and defined on open sets of the corresponding spaces. An elementary isomonodromy transformation of the first type is defined by a pair z_k, ζ_α and the eigenvector of $A_k = \mathrm{res}_{z_k} A$, corresponding to a non-zero eigenvalue λ,

$$q^T A_k = \lambda q^T \neq 0. \tag{3.7}$$

Consider the matrix

$$R = 1 + \frac{pq^T}{z - z_k}, \tag{3.8}$$

where p is the null-vector of $A(\zeta_\alpha)$ normalized so that

$$(q^T p) = z_k - \zeta_\alpha, \quad A(\zeta_\alpha)p = 0. \tag{3.9}$$

Remark. If $z_k \neq \zeta_\alpha$, then the matrix R is defined only on an open set of $\mathcal{A}_D$, where the product $(q^T p)$ of the corresponding eigenvectors is non-zero.

Equation (3.9) implies

$$R^{-1} = 1 - \frac{pq^T}{z - \zeta_\alpha}. \tag{3.10}$$

Furthermore, from the second equation (3.9) it follows, that the matrix A' given by (3.2) is regular at ζ_α. The matrix A' has a pole of rank 1 at $z_k - 1$. The rank of its residue at z_k is equal to the rank of the matrix $A_k R^{-1}(z_k)$. The left null-space of the last matrix contains the null-space of A_k and the vector q^T. Hence, the residue of A' at z_k has rank $h_k - 1$. In the same way, choosing another zero ζ_{α_2} of D and the eigenvector of $A'_k = \mathrm{res}_{z_k} A'$ corresponding to a non-zero eigenvalue, we get a matrix function A'' with a pole at z_k of rank $h_k - 2$. Further iterations give a matrix $T_k^{\alpha_1,\ldots,\alpha_{h_k}}(A)$, which is regular at z_k and has a pole of rank h_k at $z_k - 1$.

As follows from Lemma 3.2, the isomonodromy transformation $T_k^{\alpha_1,\ldots,\alpha_{h_k}}$ is uniquely defined by the choice of a pole z_k and a subset of h_k zeros ζ_{α_s} of D. These transformations are analogs of isomonodromy transformations introduced in [10] for the case of polynomial $A(z)$.

An elementary isomonodromy transformation of the second type is defined by a pair of zeros ζ_α and ζ_β of D. The corresponding matrix $R = R_{\alpha,\beta}$ is given by the formula

$$R_{\alpha,\beta} = 1 + \frac{p_\alpha q_\beta^T}{z - z_\beta - 1}, \tag{3.11}$$

where p_α and q_β are vectors defined by the equations

$$(i)\ A(\zeta_a)p_\alpha = 0;\ (ii)\ q_\beta^T A(\zeta_\beta) = 0;\ (iii)\left(q_\beta^T p_\alpha\right) = \zeta_\beta - \zeta_\alpha + 1. \tag{3.12}$$

From (3.12 iii) it follows that $R_{\alpha,\beta}^{-1} = 1 - p_\alpha q_\beta^T/(z - z_\alpha)$. Then equations (3.12 i, ii) imply that the matrix

$$T^{\alpha|\beta}(A) = R_{\alpha,\beta}^{-1}(z+1)A(z)R_{\alpha,\beta}^{-1}(z) = \left(1 + \frac{p_\alpha q_\beta^T}{z - z_\beta}\right) A(z) \left(1 + \frac{p_\alpha q_\beta^T}{z - z_\alpha}\right) \tag{3.13}$$

is regular and non-degenerate at ζ_α and ζ_β. It has the same set of poles as A. The zeros of its determinant are $\zeta_\alpha - 1,\ \zeta_\beta + 1$ and $\zeta_\gamma,\ \gamma \neq \alpha, \beta$.

The transformation $T_D^{D'}$ can be obtained as the composition of elementary isomonodromy transformations. Indeed, if D and D' are equivalent, then the poles of D can be shifted to the poles of D' by elementary transfomations (or their inverse) of the first type. After that $N-1$ zeros can be shifted to $N-1$ zeros of D' by transformations of the second type. Then, equation (3.5) defines a unique position of the last zero. The lemma is proved.

Now we are ready to present the main result of this section.

Theorem 3.1 *Let $A_0^{ij} = \rho_i \delta^{ij}$ and K be diagonal matrices and let $S(w)$ be a rational matrix function of the variable $w = e^{2\pi i z}$ having the form: (a) (2.30), if $A_0 = 1$, (b) (2.70,2.71), if $\Im \rho_i = 0$; (c) (2.74,2.75) if $\Im \ln \rho_i \neq \Im \ln \rho_j \neq 0$. Then, for each S in general position and for each set of branches z_k, ζ_α of the logarithms of poles and zeros of* $\det S(w)$, *there exists a unique rational matrix function $A(z)$ of the form (1.2) such that $S(z)$ is the connection matrix of the corresponding difference equation (1.1) and* $\det A(\zeta_\alpha) = 0$.

Proof. It has been already proven that if $A(z)$ exists for one set of z_k, ζ_α, then in general position it exists and is unique for any equivalent set. Therefore, for the proof of the theorem it is enough to construct one equation (1.1) for which S is the connection matrix.

Let us fix a real number x such that on the line $L : \Re z = x$ the matrix $S(z)$ is regular and invertible. We denote the half planes $\Re z < x$ and $\Re z > x$ by D_l and D_r, respectively. Consider the following factorization problem

Problem III. *For a given S find invertible matrix functions $\mathcal{X}_l(z)$ and $\mathcal{X}_r(z)$, which are holomorphic and bounded inside the domains $\mathcal{D}_l, D_r$, respectively, continuous up to the boundaries, and such that the functions $\Psi_{l(r)} = \mathcal{X}_{l(r)} e^{z \ln A_0 + K \ln z}$ satisfy the equation*

$$\Psi_l(\xi) = \Psi_r(\xi) S(\xi), \quad \xi \in L\,. \tag{3.14}$$

Lemma 3.4 *For a generic matrix S the Problem III has a solution which is unique up to the normalization $\mathcal{X}'_{l(r)} = g \mathcal{X}_{l(r)}$.*

Proof. Consider functions $\mathcal{X}_{l(r)}$ defined in each of the half-planes by the Cauchy integral

$$\mathcal{X}(z) = 1 + \frac{1}{2\pi i} \int_L \frac{\chi(\xi) d\xi}{\xi - z}\,. \tag{3.15}$$

Equation (3.14) is equivalent to the equation

$$\frac{1}{2} \chi(\xi)(\mathcal{M}(\xi) + 1) - \frac{1}{2\pi i} I_\chi(\xi)(\mathcal{M}(\xi) - 1) = (\mathcal{M}(\xi) - 1), \tag{3.16}$$

where $\mathcal{M} = Y_0 S Y_0^{-1},\ \ Y_0 = e^{z \ln A_0 + K \ln z}$. If S has the form (a) or (c), then at the infinity $\mathcal{M}$ exponentially tends to 1, and for a generic S equation (3.16) has a unique solution. In

the case (b) of the mild equations with real exponents, the coefficient $\mathcal{M}$ has no limit at the infinity and the fundamental results of the theory of singular integral equations can not be applied directly.

The following slight modification of the Problem III allows us to prove the lemma for the case (b). Consider functions $\mathcal{X}'_{l(r)}$ given by the Cauchy integral (3.15) over the line $\xi \in L' : Arg(\xi - x) = \pi/2 + \varepsilon$, $\varepsilon > 0$. If $\chi(\xi), \xi \in L'$, is a solution of equation (3.16) on L' with the coefficient $\mathcal{M}' = Y_0 g_r S Y_0^{-1}$, then the boundary values of the functions $\Psi' = \mathcal{X}'_l Y_0$ and $\mathcal{X}'_r Y_0$ on L' satisfy the equation

$$\Psi'_l(\xi) = \Psi'_r(\xi) g_r S(\xi), \quad \xi \in L' . \tag{3.17}$$

From (2.70) it follows that $\mathcal{M}'$ along L' exponentially tends to the identity matrix. Therefore, a solution χ of the corresponding equation (3.16) on L' exists and is unique. It defines a unique solution of the factorization problem (3.17). The equation (3.17) can be used for meromorphic extension of the functions $\Psi'_{l(r)}$, which are originally defined in the half-planes separated by L'. If $\varepsilon > 0$ is small enough, then S is regular and invertible in the sectors between L and L'. Hence, the extensions of the functions Ψ'_l and Ψ'_r are holomorphic in the domains $\mathcal{D}_l$ and D_r, respectively. Therefore, the functions $\Psi_l = \Psi'_l$ and $\Psi_r = \Psi'_r g_r$ are solutions of the factorization problem (3.14). The lemma is proved.

Let $\Psi_{l(r)}$ be a solution of the factorization problem (3.14). Then the function

$$A(z) = \Psi_l(z+1)\Psi_l^{-1}(z) = \Psi_r(z+1)\Psi_r^{-1}(z) \tag{3.18}$$

is holomorphic in the domains $\Re z < x - 1$ and $\Re z > x$. It tends to A_0 as $z \to \infty$. Inside the strip Π_{x-1} the poles of A and A^{-1} coincide with the zeros and the poles of S and S^{-1}. Therefore, $A(z)$ has the form (1.2), where $x - 1 < \Re z_m < x$. The theorem is thus proven.

4 Continuous limit

Our next goal is to show that in the continuous limit $h \to 0$ the canonical meromorphic solutions Ψ_x of difference equation (1.21) converge to solutions of differential equation (1.22).

The construction of canonical meromorphic solution Ψ_x of (1.21), which is holomorphic in the strip $\Pi_x^h : x < \Re z < x + h$, requires only slight changes of the formulas used above. As before, a sectional holomorphic solution Φ_x of the factorization problem

$$\Phi_x^+(\xi + h) = (1 + hA(\xi))\Phi_x^-(\xi), \quad \xi = x + iy \tag{4.1}$$

can be represented with the help of the Cauchy type integral

$$\Phi_x = Y_0 \phi, \quad \phi = 1 + \int_L \varphi(\xi) k_h(z, \xi)\, d\xi, \; k_h = k(h^{-1}z, h^{-1}\xi), \tag{4.2}$$

where $k(z, \xi)$ is given by (2.13) and $Y_0 = e^{z \ln(1+hA_0) + hK \ln z}$. The residue of k_h at $z = \xi$ equals h, therefore the boundary values of ϕ are

$$\phi^-(\xi) = -\frac{h\varphi(\xi)}{2} + 1 + I_\varphi(\xi), \quad \phi^+(\xi + 1) = \frac{h\varphi(\xi)}{2} + 1 + I_\varphi(\xi)\,, \tag{4.3}$$

where I_φ is the principal value of the corresponding integral. The singular integral equation for φ which is equivalent to (4.1) now takes the form

$$(2 + h\widetilde{A})\,\varphi - 2\widetilde{A}I_\varphi = 2\widetilde{A}\,, \tag{4.4}$$

where $\widetilde{A} = Y_0(\xi+1)^{-1}A(\xi)Y_0(\xi)$. If $|\,x - \Re z_k| > Ch$, then equation (4.4) can be solved by iterations. As before, the corresponding solution Ψ_x is x-independent in the intervals $\Re z_k + Ch < x < \Re z_{k+1} - Ch$. Hence, we conclude that: for any $\varepsilon > 0$, and any rational function $A(z)$ of the form (1.2) there exist h_0 such that equation (1.21) for $h < h_0$ has canonical meromorphic solutions Ψ_k, which are holomorphic in the strips $z \in \mathcal{D}_k : \Re z_k + \varepsilon < \Re z < \Re z_{k+1} - \varepsilon$.

The existence of Ψ_k implies that for each A of the form (1.2) the local monodromy matrices μ_k are well-defined for sufficiently small h. Hence, we may consider their continuous limits.

Theorem 4.1 *In the limit $h \to 0$:*

(A) *the canonical solution Ψ_k of difference equation (1.21) uniformly in $\mathcal{D}_k$ converges to a solution $\hat{\Psi}_k$ of differential equation (1.22), which is holomorphic in $\mathcal{D}_k$;*

(B) *the local monodromy matrix (1.17) converges to the monodromy of $\hat{\Psi}_k$ along the closed path from $z = -i\infty$ and goes around the pole z_k;*

(C) *the upper- and lower-triangular matrices (g_r, g_l) and (S_0, S_∞) defined in (2.70) and (2.75), respectively, for the cases of real and imaginary exponents, converge to the Stokes' matrices of differential equation (1.22).*

The first statement of the theorem follows from a simple observation, that in the continuous limit the singular integral equation for solutions of the Riemann-Hilbert factorization problem becomes just differential equation (1.22). It is easy to check that

$$k_h(z,\xi) = \begin{cases} 1 + O(h), & z - \xi > h\ln h,\ \xi > h\ln h, \\ \quad\ \ O(h), & \xi - z > h\ln h,\ \text{or}\ \xi < h\ln h, \end{cases} \quad z > h\ln h \tag{4.5}$$

Similar equations are valid for $z < h\ln h$. In both the cases we have

$$I_\varphi(z) = \int_0^z \varphi(\xi)d\xi + O(h) \tag{4.6}$$

From (4.4, 4.6) it follows that the function $\psi = 1 + I_\varphi$ satisfies the relation

$$\frac{d\psi}{dz} = A(z)\psi(z) + O(h). \tag{4.7}$$

On the line $L_x : \Re z = x$ the function Φ_x equals $\psi + 0(h)$. Therefore, Φ_x does converge to $\hat{\Psi}_k$ on L_x. For the cases of regular singular equations and mild equations with real exponents the convergence is uniform on $\mathcal{D}_k$. For the case of imaginary exponents the convergence becomes uniform only for the special choice of constant term g in the integral representation for Φ_x, which in (4.2) was set $g = 1$ (compare with (2.76)).

The second and the third statements of the theorem are direct corollaries of (A) and of the definition of the local monodromy matrices μ_k and the matrices (g_r, g_l) and (S_0, S_∞).

5 Difference equations on elliptic curves

In this section we construct direct and inverse monodromy maps for difference equations on an elliptic curve.

Let Γ be the elliptic curve with periods $(2\omega_1, 2\omega_2)$, $\Im(\omega_2/\omega_1) > 0$. Consider the equation

$$\Psi(z+h) = A(z)\Psi(z), \tag{5.1}$$

where $A(z)$ is a meromorphic $(r \times r)$ matrix function with simple poles, which satisfies the following monodromy properties

$$A(z+2\omega_\alpha) = B_\alpha A(z) B_\alpha^{-1}, \quad B_\alpha \in SL_r \tag{5.2}$$

The matrix $A(z)$ can be seen as a meromorphic section of the vector bundle $Hom(\mathcal{V}, \mathcal{V})$, where $\mathcal{V}$ is the holomorphic vector bundle on Γ defined by a pair of commuting matrices B_α. Throughout this section it is always assumed that B_α are diagonalizable. Equation (5.1) is invariant under the gauge transformation $A' = GAG^{-1}$. Therefore, if B_α are diagonalizable, then we may assume without loss of generality, that B_α are diagonal. Furthermore, if G is a diagonal matrix, then equation (5.1) is invariant under the transformation

$$\Psi' = G^z\Psi, \quad A' = G^{z+h}A(z)G^{-z}, G^{ij} = G^i\delta^{ij} \tag{5.3}$$

The matrix A' has the following monodromy properties

$$A'(z+2\omega_\alpha) = B'_\alpha A'(z)\left(B'_\alpha\right)^{-1}, \quad B'_\alpha = G^{2\omega_\alpha}B_\alpha. \tag{5.4}$$

Therefore, if B_α are diagonalizable, then we may assume without loss of generality, that

$$B_1^{lj} = \delta^{lj}, \quad B_2^{lj} = e^{\pi i q_j/\omega_1}\delta^{ij}. \tag{5.5}$$

Below we assume that $q_i \neq q_j$. Entries of the matrix A can be expressed in terms of the standard Jacobi theta-function: $\theta_3(z) = \theta_3(z|\tau)$, $\tau = \omega_2/\omega_1$. Let us define the function $\widetilde{\theta}$ by the formula

$$\widetilde{\theta}(z) = \widetilde{\theta}(z|2\omega_1, 2\omega_2) = \theta_3(z/2\omega_1|\omega_2/\omega_1). \tag{5.6}$$

The monodromy properties of θ_3 imply

$$\widetilde{\theta}(z+2\omega_1) = \widetilde{\theta}(z), \quad \widetilde{\theta}(z+2\omega_2) = -\widetilde{\theta}(z)e^{-\pi i z/\omega_1}. \tag{5.7}$$

The function $\widetilde{\theta}$ is an odd function $\widetilde{\theta}(z) = -\widetilde{\theta}(-z)$. From (5.7) it follows that the entries of A satisfying (5.2, 5.5) can be uniquely represented in the form

$$\begin{aligned} A^{ii} &= \rho_i + \sum_{m=1}^n A_m^i \widetilde{\zeta}(z-z_m), \quad \sum_m A_m^i = 0, \\ A^{ij} &= \sum_{m=1}^n A_m^{ij} \frac{\widetilde{\theta}(z-q_i+q_j-z_m)}{\widetilde{\theta}(z-z_m)}, \quad i \neq j, \end{aligned} \tag{5.8}$$

where $\widetilde{\zeta} = \partial_z(\ln \widetilde{\theta})$, and $z_m \in C$ are the poles of $A(z)$ in the fundamental domain

$$0 < r(z_m) < 1, \quad 0 < u(z_m) < 1 \tag{5.9}$$

of C/Λ, $\Lambda = \{2n\omega_1, 2m\omega_2\}$. Here and below we will use the notation $r(z)$ and $u(z)$ for real coordinates $z = 2r\omega_1 + 2u\omega_2$ of $z \in C$ with respect to the basis $2\omega_\alpha$,

$$r(z) = \frac{z\bar{\omega}_2 - \bar{z}\omega_2}{2(\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2)}, \; u(z) = \frac{z\bar{\omega}_1 - \bar{z}\omega_1}{2(\omega_2\bar{\omega}_1 - \bar{\omega}_2\omega_1)}, \tag{5.10}$$

Throughout this section it is assumed that the poles z_m of A are non congruent (mod h), i.e. $h^{-1}(z_m - z_k) \notin Z$.

Our goal is to construct canonical meromorphic solutions of equation (5.1) with the coefficients of the form (5.8). As before, this problem is reduced to a proper Riemann-Hilbert factorization problem. For definiteness we assume that the step h of the difference equation satisfies the condition

$$0 < r(h) < 1. \tag{5.11}$$

Let us fix a real number x and consider the following problem in the strip $z \in \Pi_x : \; x \leq r(z) \leq x + r(h)$.

Problem IV. *Find in the strip Π_x a continuous matrix function $\Phi(z)$, which is meromorphic inside Π_x, and whose boundary values on two sides of the strip satisfy the equation*

$$\Phi^+(\xi + h) = A(\xi)\Phi^-(\xi), \quad r(\xi) = x. \tag{5.12}$$

The index of the problem is given by the integral

$$ind_x(A) = \int_{L_x} d \ln \det A, \; \xi \in L_x : r(\xi) = x, \tag{5.13}$$

Lemma 5.1 *For a generic $A(z)$, such that $ind_x(A) = 0$, there exists a non-degenerate holomorphic solution Φ_x of the problem (5.12) having the following monodromy property*

$$\Phi_x(z + 2\omega_2) = e^{\pi i \hat{q}/\omega_1}\Phi_x(z)e^{-2\pi i \hat{s}}, \tag{5.14}$$

where $\hat{q}$ is the diagonal matrix defining the monodromy property (5.2,5.5) of A, and $\hat{s}$ is a diagonal matrix $\hat{s}^{ij} = s^i\delta^{ij}$. The solution Φ_x is unique up to the transformation $\Phi'_x = \Phi_x F$, where F is diagonal.

Proof. The lemma can be proved by methods of algebraic geometry. Indeed, let us define an action of the lattice Λ_h span by h and $2\omega_2$ on $(z, f) \in C \times C^r$ as follows:

$$(z, f) \to (z + h, A(z)f), \quad (z, f) \to (z + 2\omega_2, B_2 f), \quad B_2 = e^{\pi i \hat{q}/\omega_1}. \tag{5.15}$$

Then the factor-space $C \times C^r/\Lambda_h$ is a vector-bundle $\mathcal{V}$ on the elliptic curve Γ_h with periods $(h, 2\omega_2)$. From (5.13) it follows that the determinant bundle of $\mathcal{V}$ is of degree zero, $c_1(\mathcal{V}) =$

0. According to [18], for a generic zero degree vector bundle on an algebraic curve there exists a flat holomorphic connection. A basis of horizontal sections of such connection can be identified with a holomorphic matrix function Φ' satisfying the relations $\Phi'(z+h) = A(z)\Phi'(z)V_1$, $\Phi'(z+2\omega_2) = B_2\Phi'(z)V_2$, where V_1, V_2 is a pair of commuting matrices. The change of the basis of horizontal sections corresponds to the transformation $\Phi' \to \Phi g$, $V_i \to g^{-1}V_i g$. Therefore, in the general position when V_i are diagonalizable, we may assume, without loss of generality, that V_i are diagonal. Now we can define a holomorphic solution of the boundary problem (5.12) as follows $\Phi_x = \Phi' V_1^{-z/h}$. It satisfies the monodromy relation (5.14), where $e^{-2\pi\hat{s}/h} = V_2 V_1^{-2\omega_2/h}$. We call Φ_x the *Bloch solution* of the factorization problem (5.12). In the general position we may assume that $s_i \neq s_j$.

Suppose, that there are two Bloch solutions Φ_x and Φ'_x of the factorization problem (5.12). From (5.13) it follows that Φ_x is non-degenerate in Π_x. Therefore, the entries of the matrix function $F = \Phi_x^{-1}\Phi'_x$ are holomorphic matrix functions satisfying the relations

$$F^{lj}(z+h) = F(z), \quad F^{lj}(z+2\omega_2) = F^{lj}(z)e^{2\pi i(s_l - s'_j)/h} \tag{5.16}$$

Equations (5.16) imply that $F^{ij} = 0$, if $s_i \neq s_j$. Indeed, consider the function

$$\hat{F}^{ij} = F^{ij}\widetilde{\theta}_h(z + s_i - s'_j)/\widetilde{\theta}_h(z), \tag{5.17}$$

where $\widetilde{\theta}_h$ is the function given by the formula (5.7) for the Γ_h, i.e.

$$\widetilde{\theta}_h(z) = \widetilde{\theta}(z|h, 2\omega_2). \tag{5.18}$$

From (5.16) it follows, that $\hat{F}^{ij}$ is an elliptic function on Γ_h with one simple pole at $z = 0$. There is no such a non-trivial function. Hence, $s_i = s'_i$ and $F^{ij} = 0, i \neq j$, and the Lemma is thus proven.

Now we are ready to define the direct monodromy map for difference equations (5.1) with coefficients A of the form (5.8). As before, a holomorphic solution Φ_x of the boundary problem (5.12) defines a meromorphic solution $\Psi_x(z)$ of (5.1). From (5.14) it follows that it satisfies the Bloch relation (1.25).

The matrix A has period $2\omega_1$. That implies

$$\Phi_{x+1}(z + 2\omega_1) = \Phi_x(z), \quad z \in \Pi_x\,. \tag{5.19}$$

Therefore, the matrix $\Psi_x(z - 2\omega_1)$ is the Bloch solution of (5.1), which is holomorphic in the strip Π_{x+1}. Let us consider the connection matrix of two Bloch solutions

$$S_x(z) = \Psi_x^{-1}(z - 2\omega_1)\Psi_x(z). \tag{5.20}$$

For obvious reason the matrix S_x is h-periodic. Let us show that it has the following monodromy properties

$$S(z+h) = S(z), \quad S(z+2\omega_2) = e^{2\pi i\hat{s}/h}S(z)e^{-2\pi i\hat{s}/h}, \tag{5.21}$$

where $\hat{s}$ is the diagonal matrix defined by the monodromy properties (5.14) of Φ_x.

By definition the connection matrix S_x depends on the choice of x. Let us fix $x = 0$, and denote $S_{x=0}(z)$ by $S(z)$.

Theorem 5.1 *In the general position the entries of the monodromy matrix $S(z)$ have the form*

$$\begin{aligned} S^{ii} &= S_0^i + \sum_{m=1}^n S_m^i \zeta_h(z-z_m), \ \sum_{m=1}^n S_m^i = 0, \\ S^{ij} &= \sum_{m=1}^n S_m^{ij} \frac{\widetilde{\theta}_h(z-s_i+s_j-z_m)}{\widetilde{\theta}_h(z-z_m)}, \ i \neq j. \end{aligned} \tag{5.22}$$

where $\zeta_h = \partial_z \ln \widetilde{\theta}_h$, and $\widetilde{\theta}_h$ are given by (5.18).

Recall, that z_m are the poles of $A(z)$ in the fundamental domain (5.9) of C/Λ.

Proof. In the half-plane $r(z) > 0$ the function $\Psi_{x=0}$ has poles at the points $z_m + nh + 2m\omega_2, n = 1, 2, \ldots, \ m \in Z$. By definition, the function $\Psi_{x=1}$ is holomorphic in $\Pi_{x=1}$. Therefore, in the strip Π_1 the matrix S has the poles at the points congruent to $z_m \bmod \Lambda_h$. Then equations (5.21) imply (5.22).

We refer to above defined correspondence

$$\{\rho_i, A_m^{ij}, q_i\} \longmapsto \{S_0^i, S_m^{ij}, s_i\} \tag{5.23}$$

as the direct monodromy map.

5.1 Local monodromies

All the results that were obtained above for the case of difference equations with rational coefficients have analogs in the elliptic case. For example, the analogues of the special regular singular equations are equations (5.1) with coefficients $A(z)$ such that their residues A_m of $A(z)$ are rank 1 matrices, and the determinant of A identically equals 1, $\det A(z) = 1$, and such that the parameters q_i in (5.8) satisfy the constraint

$$\sum_{i=1}^r q_i = 0. \tag{5.24}$$

The space of such matrices will be denoted by $\mathcal{A}_0(\Gamma)$. The dimension of $\mathcal{A}_0(\Gamma)$ equals $\dim \mathcal{A}_0(\Gamma) = n(2r-1) - n + (r-1) = (2n+1)(r-1)$. The first term in the last equation is the dimension of the subspace of matrix functions of the form (5.8) having rank 1 residues. The second term is the number of conditions equivalent to the constraint $\det A = 1$. The last term is the number of parameters q_i. Let $\mathcal{B}(\Gamma) = \mathcal{A}_0(\Gamma)/C^{r-1}$ be the quotient of $\mathcal{A}_0(\Gamma)$ under the action $A \to gAg^{-1}$, where g is the diagonal matrix. The dimension of $\mathcal{B}(\Gamma)$ equals $\dim \mathcal{B}(\Gamma) = 2n(r-1)$. Explicit parameterization of an open set of the space $\mathcal{B}(\Gamma)$ can be obtained as follows. Let us order the poles, and consider matrices $A(z)$ of the form

$$A(z) = L_n(z) L_{n-1}(z) \cdots L_1(z) \tag{5.25}$$

where

$$L_m^{ij} = f_m^i \; \frac{\widetilde{\theta}(z - q_{i,m+1} + q_{j,m} - z_m)}{\widetilde{\theta}(z - z_m)\; \widetilde{\theta}(q_{i,m+1} - q_{j,m})}, \tag{5.26}$$

and $q_{i,m}$ are complex numbers satisfying (5.24) and such that $q_{i,n+1} = q_{i,1}$.

The residue of L_m at z_m has rank 1. Therefore, its determinant has at most simple pole at z_m. The constraint (5.24) for $q_{i,m}$ implies that $\det L_m$ is an elliptic function. Therefore, it is constant. The vector f_m can be normalized by the condition $\det L_m(z) = \det L(0) = 1$

$$\prod_{i=1}^{r} f_i^{-1} = \det \left[\frac{\widetilde{\theta}(z_m + q_{i,m+1} - q_{j,m})}{\widetilde{\theta}(z_m)\; \widetilde{\theta}(q_{i,m+1} - q_{j,m})} \right]. \tag{5.27}$$

The number of parameters $(f_{i,m}, q_{i,m})$ in (5.25) satisfying the constraints (5.24) and (5.27) equals the dimension of $\mathcal{B}(\Gamma)$.

Let us assume, that the first coordinates $r_m = r(z_m)$ of the poles of A in the basis $2\omega_\alpha$ are distinct $r_l < r_m,\; l < m$. Below we use the notations $r_0 = 0,\; r_{n+1} = 1$.

Theorem 5.2 *For a generic matrix $A \in \mathcal{A}_0(\Gamma)$ the equation (5.1) has a unique set of meromorphic solutions $\Psi_k,\; k = 0, \ldots, n$, which are holomorphic in the strips $r_k < r(z) < r_{k+1} + r(h)$ and satisfy the relation*

$$\Psi_k(z + 2\omega_2) = e^{\pi i \hat{q}/\omega_1} \Psi_k(z) e^{-2\pi i \hat{s}_k/h},\; \hat{s}_k^{ij} = s_{i,k}\delta^{ij}, \tag{5.28}$$

and such that the local connection matrices $M_k = \Psi_k^{-1}\Psi_{k-1},\; k = 1, \ldots, n$, have the form

$$M_k = \alpha_{i,k} \; \frac{\widetilde{\theta}_h(z - s_{i,k} + s_{j,k-1} - z_k)}{\widetilde{\theta}_h(z - z_m)\widetilde{\theta}_h(s_{i,k} - s_{j,k-1})}, \tag{5.29}$$

where $s_{i,k}$ and $\alpha_{i,k}$ satisfy the relations

$$\sum_{i=1}^{r} s_{i,k} = 0, \quad \prod_{i=1}^{r} \alpha_{i,k}^{-1} = \det \left[\frac{\widetilde{\theta}_h(z_k + s_{i,k} - s_{j,k-1})}{\widetilde{\theta}_h(z_k)\widetilde{\theta}_h(q_{i,k} - q_{j,k-1})} \right]. \tag{5.30}$$

The map $\{f_m^i, q_{i,m}\} \longmapsto \{\alpha_k^i, s_{i,k}\}$ is a one-to-one correspondence of open sets of the varieties defined by the constraints (5.24, 5.27) and (5.30), respectively.

Proof. The existence of a meromorphic solution Ψ_k', which is holomorphic in the strip $r_k < r(z) < r_{k+1} + r(h)$ and satisfies the relation (5.28) follows from the Lemma 5.1. The matrix $M_k' = (\Psi_k')^{-1}\Psi_{k-1}'$ has period h, i.e. $M_k'(z + h) = M_k'(z)$. From (5.28) it follows that

$$M_k'(z + 2\omega_2) = e^{2\pi i \hat{s}_k/h} M_k'(z) e^{-2\pi i \hat{s}_{k-1}/h}.$$

In the strip $\Pi_{r_k + r(h)}$ it has simple poles at the point z_k, where its residue has rank 1. Therefore, *a'priory* it can be represented in the form

$$M_k' = \widetilde{\alpha}_{i,k}\beta_{j,k} \; \frac{\widetilde{\theta}_h(z - s_{i,k} + s_{j,k-1} - z_k)}{\widetilde{\theta}_h(z - z_k)\widetilde{\theta}_h(s_{i,k} - s_{j,k-1})}, \tag{5.31}$$

The solutions Ψ'_k are unique up to the transformation $\Psi'_k = \Psi_k F_k$, where F_k is a diagonal matrix $F_k^i \delta^{ij}$. If we set $F_{k-1}^j = \beta_{j,k}$, then the corresponding matrix $M_k = F_k^{-1} M'_k F_{k-1}$ has the form (5.29). The constraint (5.30) is equivalent to the equation $\det M_k = 1$.

The proof of the last statement of the theorem is reduced to the Riemann-Hilbert problem on a set of lines $r(z) = r_m + \varepsilon$. The solvability of the corresponding problem for a generic set of matrices M_k follows from the Riemann-Roch theorem.

Remark. Elliptic analog of the unitary equations considered in the Section 2 can be defined for the case of real elliptic curves. A generalization of the corresponding results obtained above for the rational case is straightforward.

5.2 Isomonodromy transformations.

The characterization of equations (5.1) on Γ having the same monodromy data is a straightforward generalization of the corresponding results in the rational case.

From (5.2) it follows, that the determinant of $A \in \mathcal{A}(\Gamma)$ is an elliptic function

$$\det A(z) = D(z) = c \frac{\prod_{\alpha=1}^N \widetilde{\theta}(z - \zeta_\alpha)}{\prod_{k=1}^n \widetilde{\theta}(z - z_k)^{h_k}}, \quad \sum_{\alpha=1}^N \zeta_\alpha = \sum_{k=1}^n h_k z_k, \quad N = \sum_k h_k. \tag{5.32}$$

As before, we denote the subspace of matrix functions having fixed determinant $D(z)$ by $\mathcal{A}_D(\Gamma) \subset \mathcal{A}(\Gamma)$.

Lemma 5.2 *(i) Two matrix functions $A(z)$ and $A'(z)$ of the form (5.8) under the map (5.23) correspond to the same connection matrix $S(z)$ if and only if they are related by the equation*

$$A'(z) = R(z+1) A(z) R^{-1}(z), \tag{5.33}$$

where the matrix R has the following monodromy properties

$$R(z + 2\omega_1) = R(z), \;\; R(z + 2\omega_2) = e^{\pi i \hat{q}'/\omega_1} R(z)\, e^{-\pi i \hat{q}/\omega_1}. \tag{5.34}$$

(ii) If the zeros ζ_α are not congruent, i.e. $(\zeta_\alpha - \zeta_\beta) h^{-1} \notin Z$, then the monodromy correspondence (5.23) restricted to $\mathcal{A}_D(\Gamma)$ is injective.

The proof of the lemma follows directly form the definition of $S(z)$ and the monodromy properties of the canonical solutions of difference equations.

Let us call the two elliptic functions D and D' equivalent, if the sets of their poles z_i, z'_i and zeros $\zeta_\alpha, \zeta'_\alpha$ are congruent *mod* h to each other, i.e $(z_i - z'_i) h^{-1} \in Z, \;\; (\zeta_\alpha - \zeta'_\alpha) h^{-1} \in Z$.

Theorem 5.3 *For each pair of equivalent elliptic functions D and D' there exists a unique isomonodromy transformation*

$$T_D^{D'}(\Gamma) : \mathcal{A}_D(\Gamma) \longmapsto \mathcal{A}_{D'}(\Gamma) \tag{5.35}$$

Proof. Let $A(z) \in \mathcal{A}_D$ be a matrix of the form (5.8). An elementary isomonodromy transformation of the first type is defined by a pair z_m, ζ_α and the left eigenvector v of $A_m = \mathrm{res}_{z_m} A$, corresponding to a non-zero eigenvalue λ (see (3.7)

Consider the matrix $R(z)$ such that the entries of the inverse matrix have the form

$$\left(R^{-1}\right)^{ij} = p^i \ \frac{\widetilde{\theta}(z - q_i + q_j' - \zeta_\alpha)}{\widetilde{\theta}(z - \zeta_\alpha)}, \tag{5.36}$$

where p^i are coordinates of the the null-vector of $A(\zeta_\alpha)$,

$$A(\zeta_\alpha) p = 0. \tag{5.37}$$

The residue of R^{-1} at ζ_α) has rank 1. Therefore, the determinant of R^{-1} has one simple pole at ζ_α. If the parameters q_i' satisfy the condition

$$\sum_{i=1}^{r} q_i' = \zeta_\alpha - z_m + \sum_{i=1}^{m} q_i, \tag{5.38}$$

then $\det R^{-1}$ has a zero at z_m. In the general position the parameters q_j' are uniquely defined by (5.38) and the equation

$$v R^{-1}(z_m) = 0. \tag{5.39}$$

Equation (5.39) implies that the matrix R has the form:

$$R^{ij} = v^j \ \frac{\widetilde{\theta}(z - q_i' + q_j - z_m)}{\widetilde{\theta}(z - z_m)}, \tag{5.40}$$

Consider now the matrix A' given by (5.33). From (5.37) it follows that A' is regular at ζ_α. The matrix A' has a pole of rank 1 at $z_m - 1$. The rank of its residue at z_m equals the rank of the matrix $A_m R^{-1}(z_m)$. The left null-space of the last matrix contains the null-space of A_m and the vector v. Hence, the residue of A' has rank $h_m - 1$. As in the rational case, further iterations give a matrix $T_i^{\alpha_1,\ldots,\alpha_{h_i}}(A)$, which is regular at z_m and has a pole of rank h_m at $z_m - h$.

As follows from Lemma 5.3, the isomonodromy transformation $T_m^{\alpha_1,\ldots,\alpha_{h_m}}$ is uniquely defined by the choice of a pole z_m and a subset of h_m zeros ζ_{α_s} of D.

An elementary isomonodromy transformation of the second type is defined by a pair of zeros ζ_α and ζ_β of D. Let v_α and v_β be the corresponding null-vectors, i.e.

$$A(\zeta_a) v_\alpha = 0; \quad v_\beta^T A(\zeta_\beta) = 0. \tag{5.41}$$

Then the same arguments as above, show that there exists a unique, up to a constant factor, matrix $R = R_{\alpha,\beta}$ of the form

$$R_{\alpha,\beta}^{ij} = v_\beta^j \ \frac{\widetilde{\theta}(z - q_i^{\alpha,\beta} + q_j - \zeta_\beta - h)}{\widetilde{\theta}(z - \zeta_\beta - h)}, \tag{5.42}$$

and such that

$$\left(R^{-1}_{\alpha,\beta}\right)^{ij} = v^i_\alpha \, \frac{\widetilde{\theta}_1(z - q_i + q_j^{\alpha,\beta} - \zeta_\alpha)}{\widetilde{\theta}(z - \zeta_\alpha)}, \tag{5.43}$$

Equations (5.41) imply that the matrix $T^{\alpha|\beta}(A) = R^{-1}_{\alpha,\beta}(z+h)A(z)R^{-1}_{\alpha,\beta}(z)$ is regular and non-degenerate at ζ_α and ζ_β. It has the same set of poles as A. The zeros of its determinant are $\zeta_\alpha - h,\ \zeta_\beta + h$ and $\zeta_\gamma,\ \gamma \neq \alpha, \beta$.

The transformation $T_D^{D'}(\Gamma)$ can be obtained as a composition of elementary isomonodromy transformations. The theorem is thus proven.

Isomonodromy deformations changing elliptic curves. The isomonodromy transformations $T_D^{D'}(\Gamma)$ are analogs of the isomonodromy transformations constructed in Section 3 for difference equations with rational coefficients. In the elliptic case there exist isomonodromy transformations which have no analog in the rational case for the obvious reason: they change the periods of the corresponding elliptic curves.

Our next goal is to define an elementary isomonodromy transformation of the third kind which keeps the poles of A and zeros of its determinant fixed.

Lemma 5.3 *For a generic matrix function $A(z)$ of the form (5.8) there exists a meromorphic matrix function $\mathcal{R}(z)$, which is holomorphic in the strip $\Pi_* : 0 < r(z) < 1 + r(h)$ and satisfies the following monodromy relations*

$$\mathcal{R}(z + 2\omega_1 + h)A(z) = \mathcal{R}(z), \quad \mathcal{R}(z + 2\omega_2) = e^{2\pi i \hat{q}'/(2\omega_1+h)}\mathcal{R}(z)e^{-\pi i \hat{q}/\omega_1}, \tag{5.44}$$

where $\hat{q}'$ is diagonal. The function $\mathcal{R}$ is unique up to the transformation $\mathcal{R}' = F\mathcal{R}$, where $F \in GL_r$ is a diagonal matrix.

The function $\mathcal{R}$ satisfying the relations (5.44) can be regarded as the canonical Bloch solution of difference equation (1.29). Its existence can be proved along with the lines identical to that used in the proof of the Lemma 5.1.

Consider now the matrix function $A' = \mathcal{R}(z+h)A(z)\mathcal{R}^{-1}(z)$. From (5.44) it follows that

$$A'(z + 2\omega_1 + h) = A'(z), \quad A'(z + 2\omega_1) = e^{2\pi i \hat{q}'/(2\omega_1+h)}A'(z)e^{-2\pi i \hat{q}'/(2\omega_1+h)}. \tag{5.45}$$

Suppose, that the matrix A is holomorphic and invertible in the strip $\Pi_{x=0}$. Then A' in the fundamental parallelogram, corresponding to the elliptic curve with periods $(2\omega_1 + h, 2\omega_2$ has the same poles z_m as A. In this parallelogram the zeros ζ_α of its determinant coincide with the zeros of $\det A$.

Remark. If the conditions $r(h) < r(z_m),\ r(h) < r(\zeta_a)$ are not satisfied, then an extra pole (or zero of the determinant) of A' in $\Pi_{x=1}$ is congruent $(mod h)$ to the pole (or zero of the determinant) of A' in Π_0.

Theorem 5.4 *If the matrix A is invertible in Π_0, then the above defined transformation $A' = \mathcal{R}(z+h)A(z)\mathcal{R}^{-1}(z)$ is isomonodromic.*

For the proof of the theorem it is enough to note, that under the assumption of the theorem the canonical Bloch solution Ψ_1 of (5.1) is holomorphic and invertible in the strip $\Pi_{1+r(h)}$. Therefore, the Bloch solutions of equation (5.1) with the coefficient A', which define the connection matrix S' are equal to

$$\Psi'_{x=0} = \mathcal{R}\Psi_0, \quad \Psi'_{1+r(h)} = \mathcal{R}\Psi_1. \tag{5.46}$$

Hence, $S'(z) = S(z)$.

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